Lecture 10

Interval Trees

Source: Introduction to Algorithms, CLRS

Interval

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Example: $[5,8]$ and $[6,9]$ are overlapping. $[3,5]$ and $[7,10]$ are non-overlapping.

Testing Overlapping of Intervals

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Overlapping intervals

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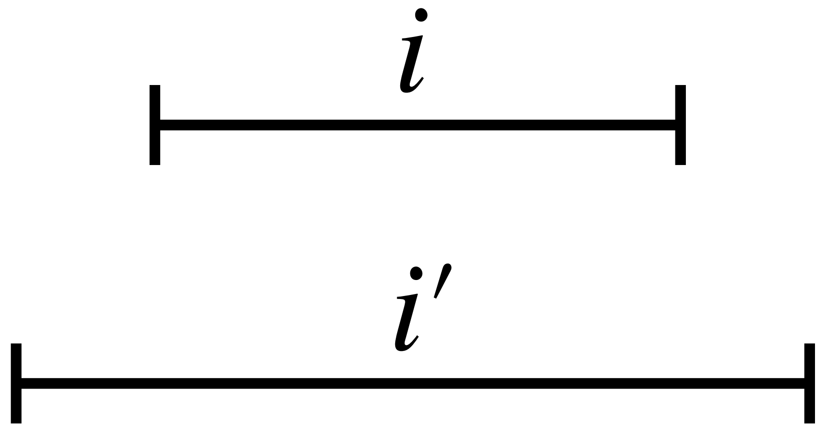

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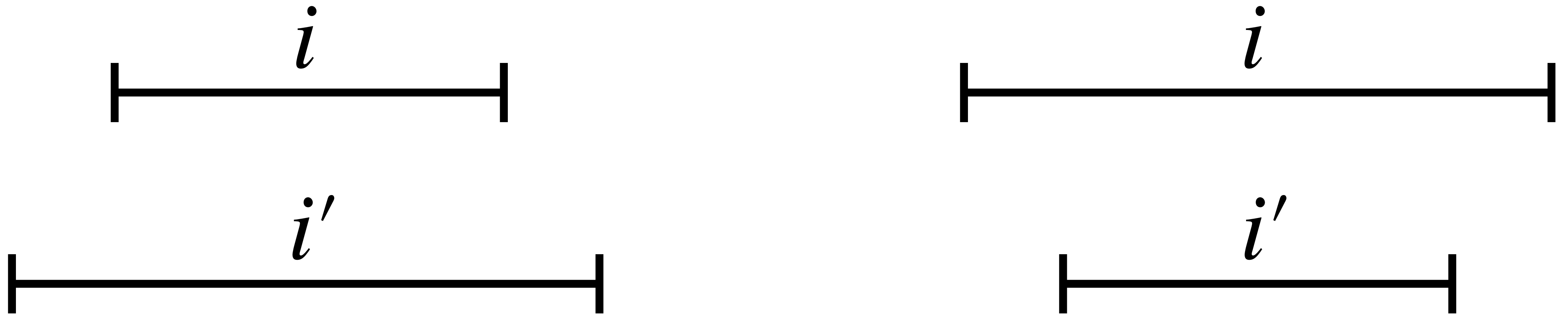

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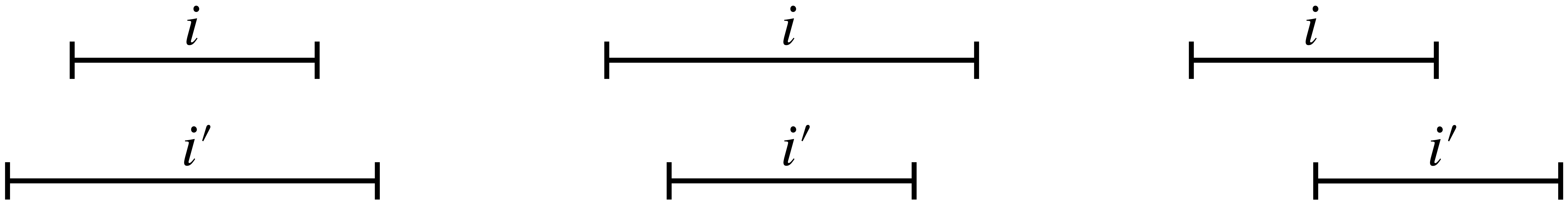

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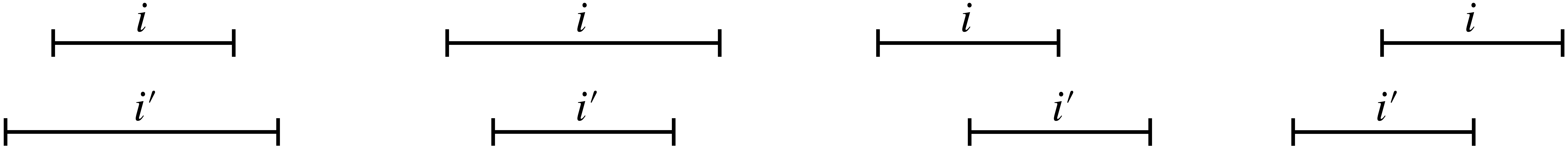

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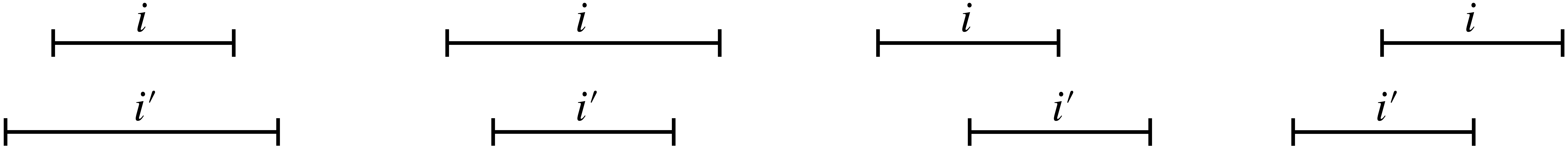

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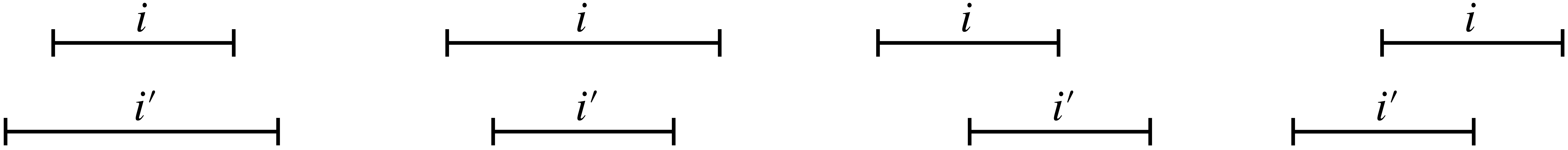

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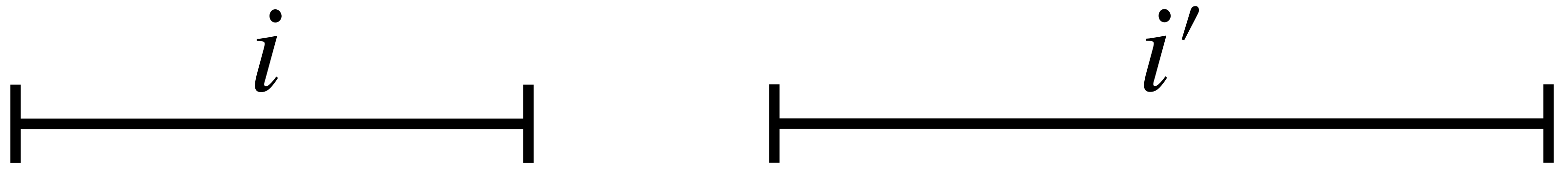

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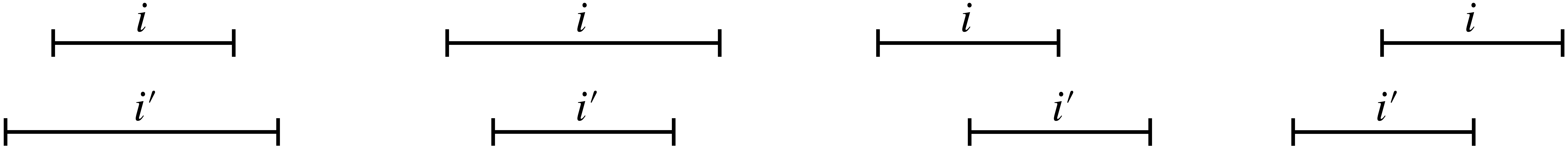

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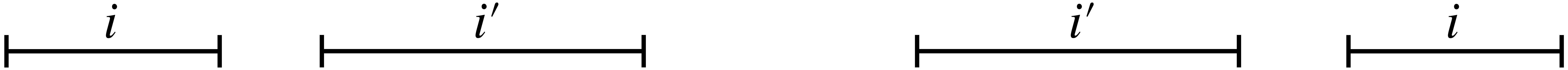
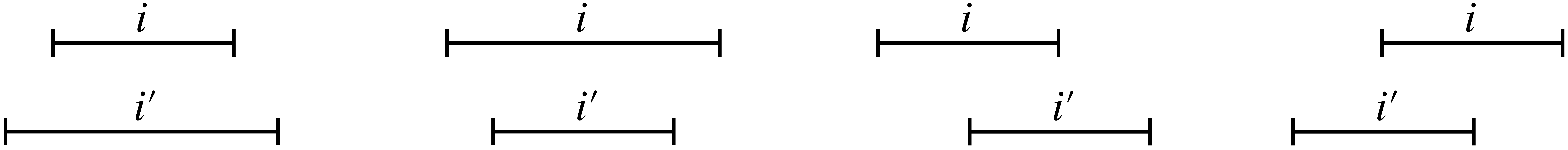


Non-overlapping intervals

Testing Overlapping of Intervals

Defn: Two intervals i and i' do not overlap if and only if $i\,.\,high < i'\,.\,low$ or $i'\,.\,high < i\,.\,low$.



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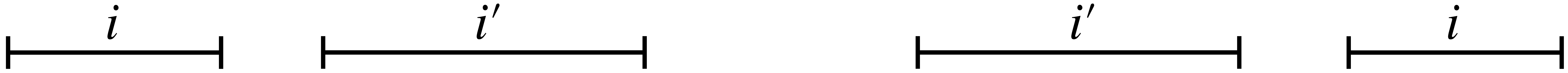

Non-overlapping intervals

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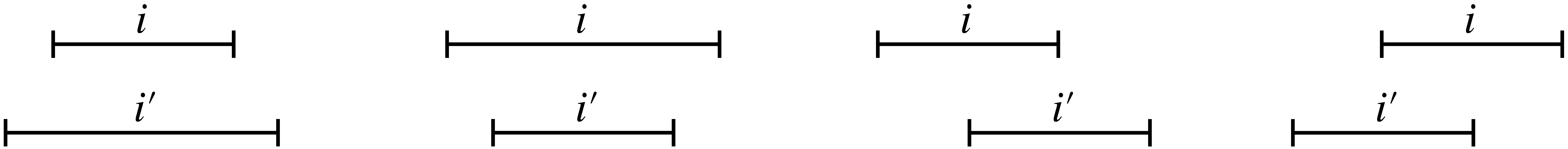

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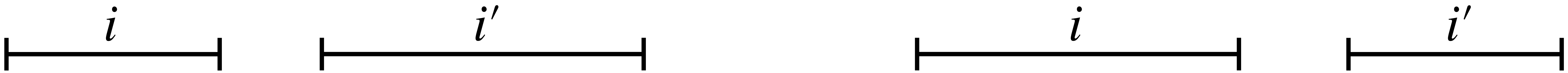

Non-overlapping intervals

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- Interval-Insert(T, x)
- Interval-Delete(T, x)
- Interval-Search(T, i)

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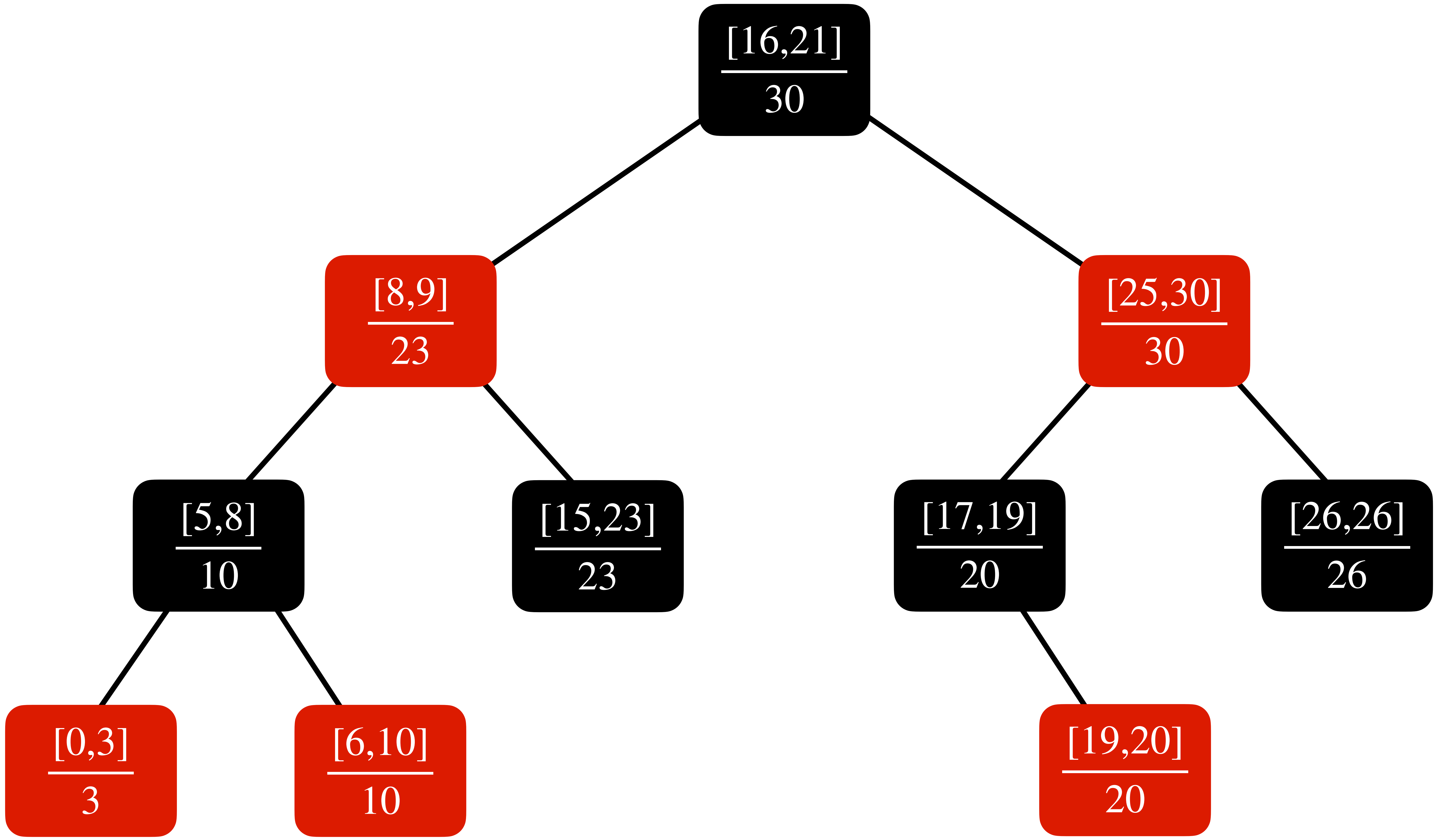
- Interval-Insert(T, x)
- Interval-Delete(T, x)
- Interval-Search(T, i)

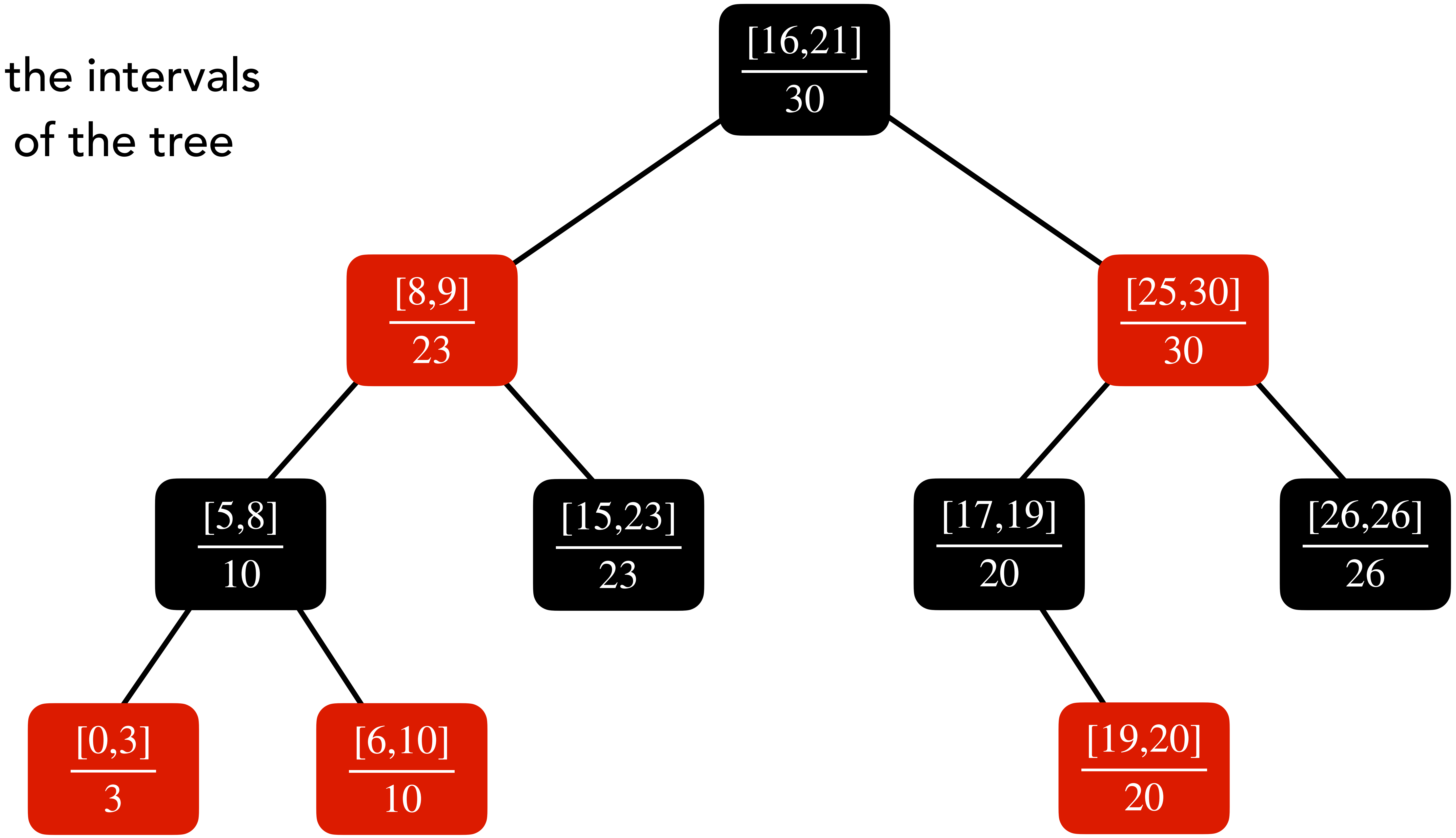
Returns an element x of T such that $x\,.\,int$ overlaps with i.

Example of an Interval Tree



Example of an Interval Tree

low values of the intervals are the keys of the tree



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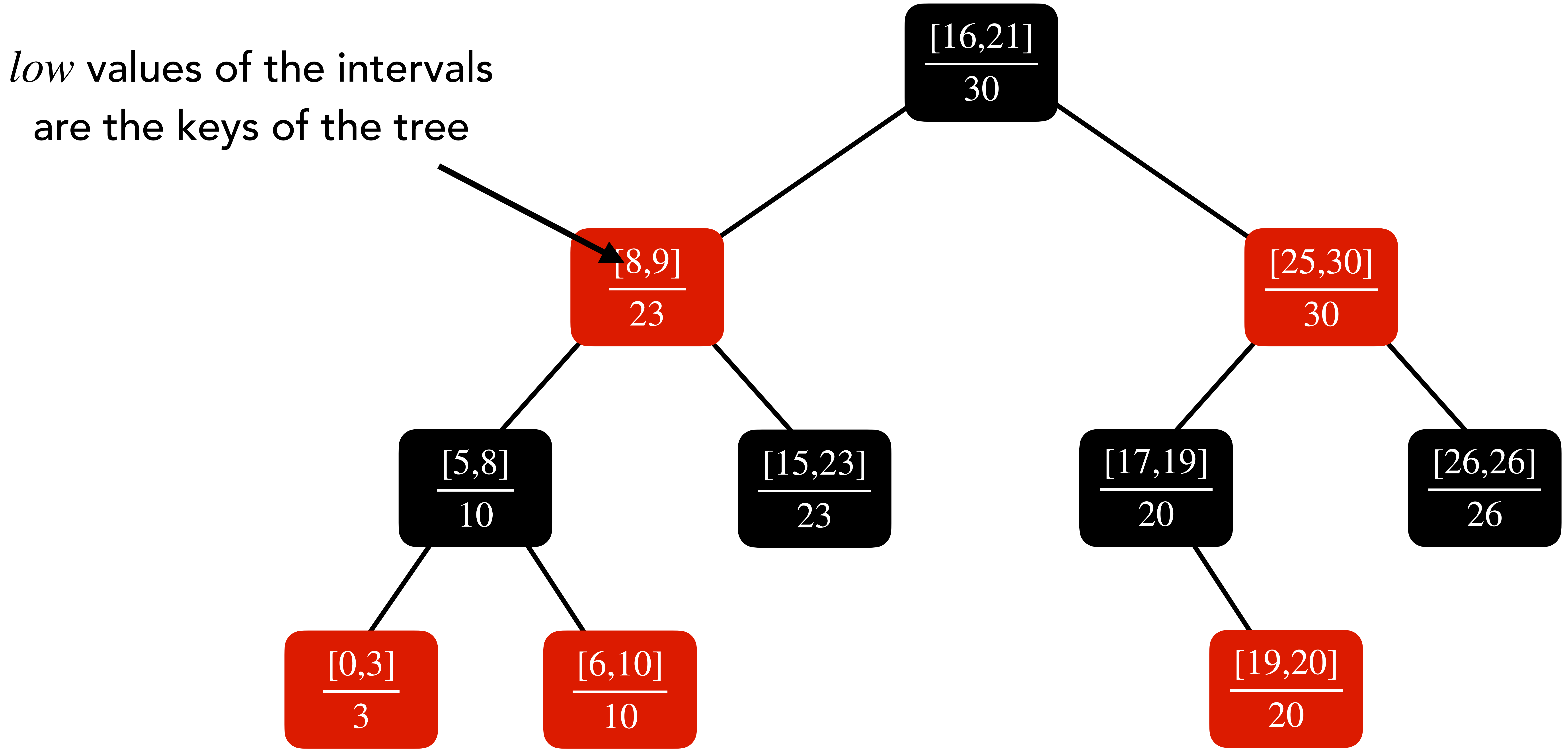

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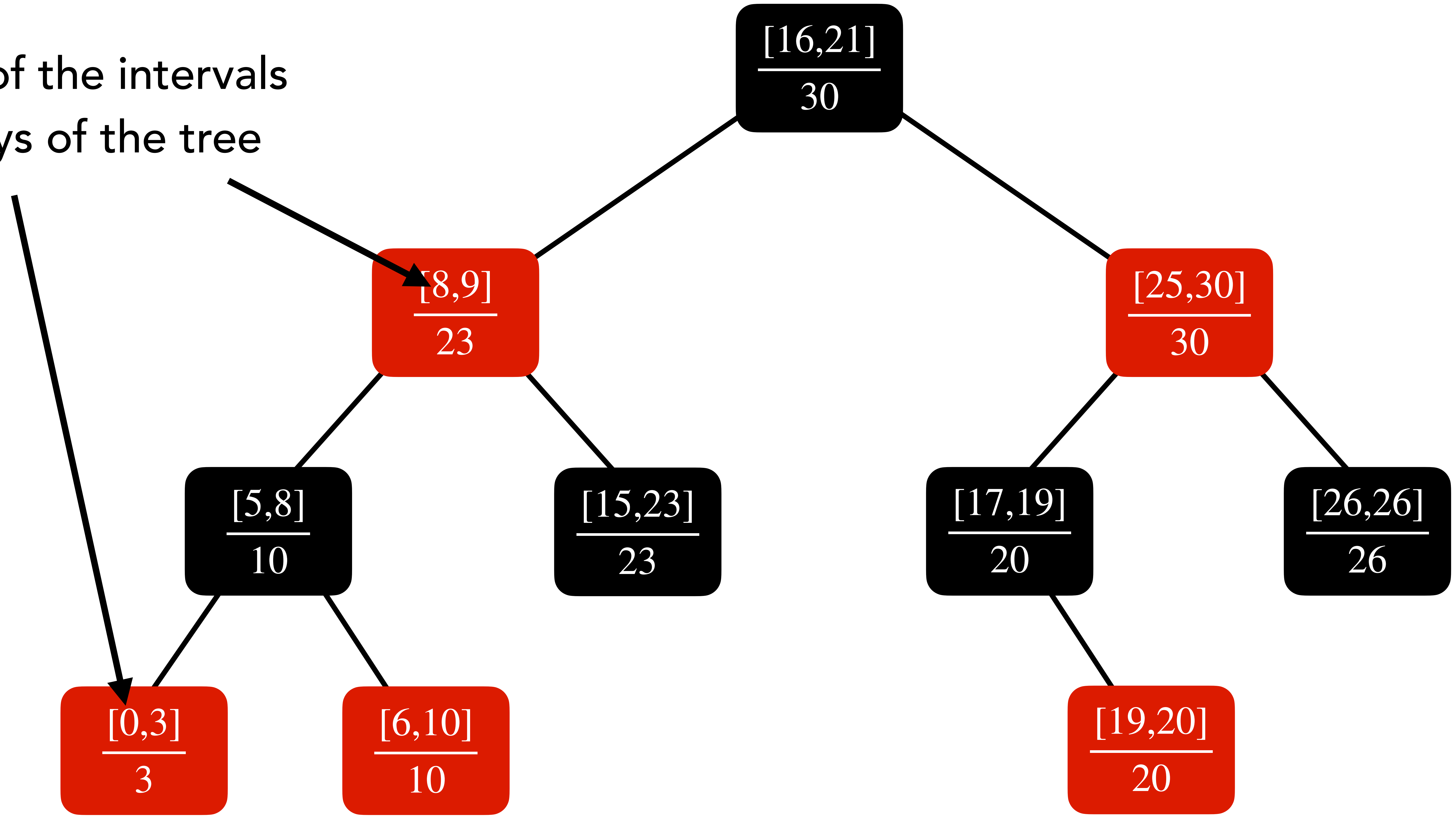

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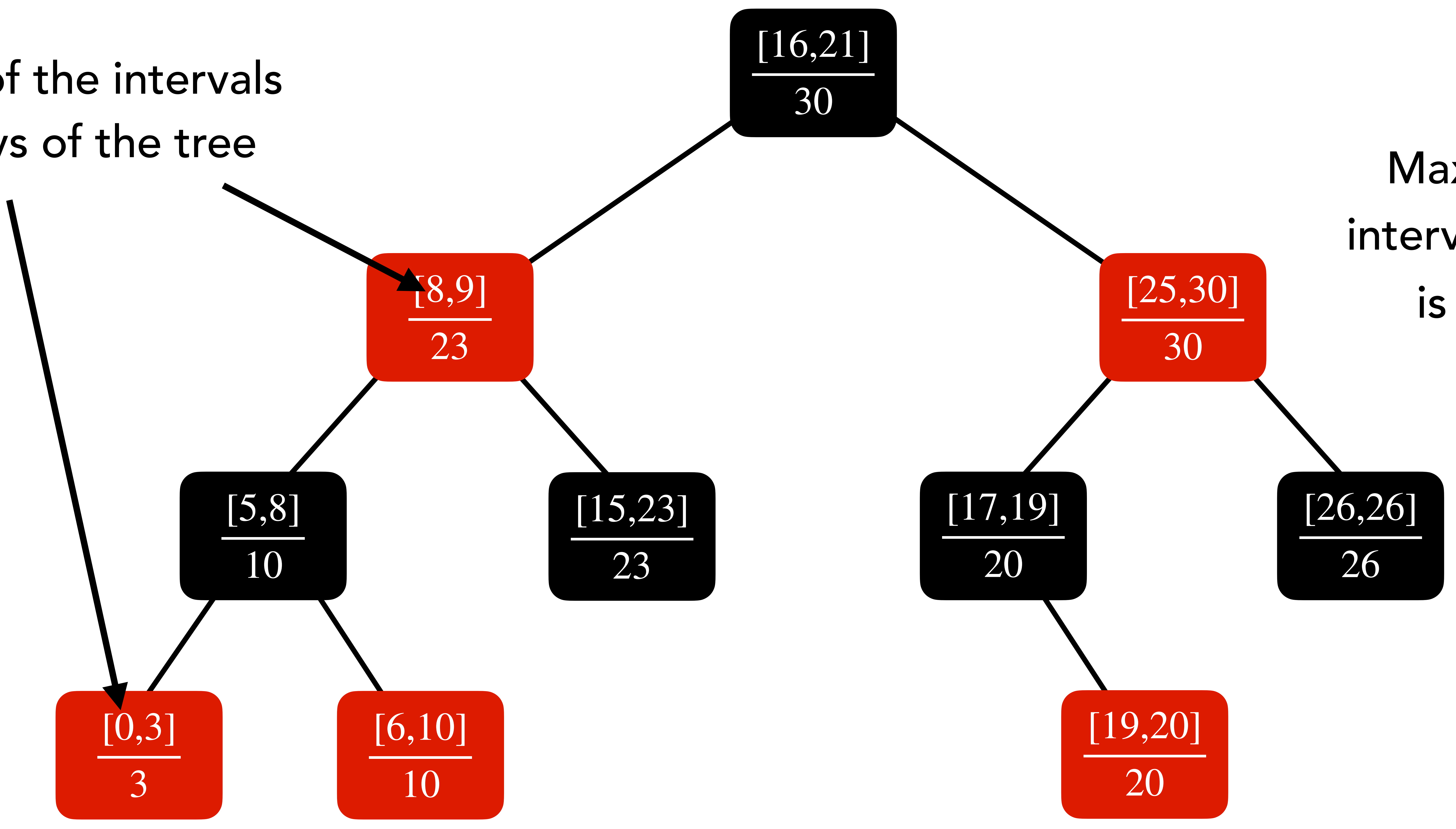

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Max high value of the intervals in the subtree(x) is stored as $x.max$

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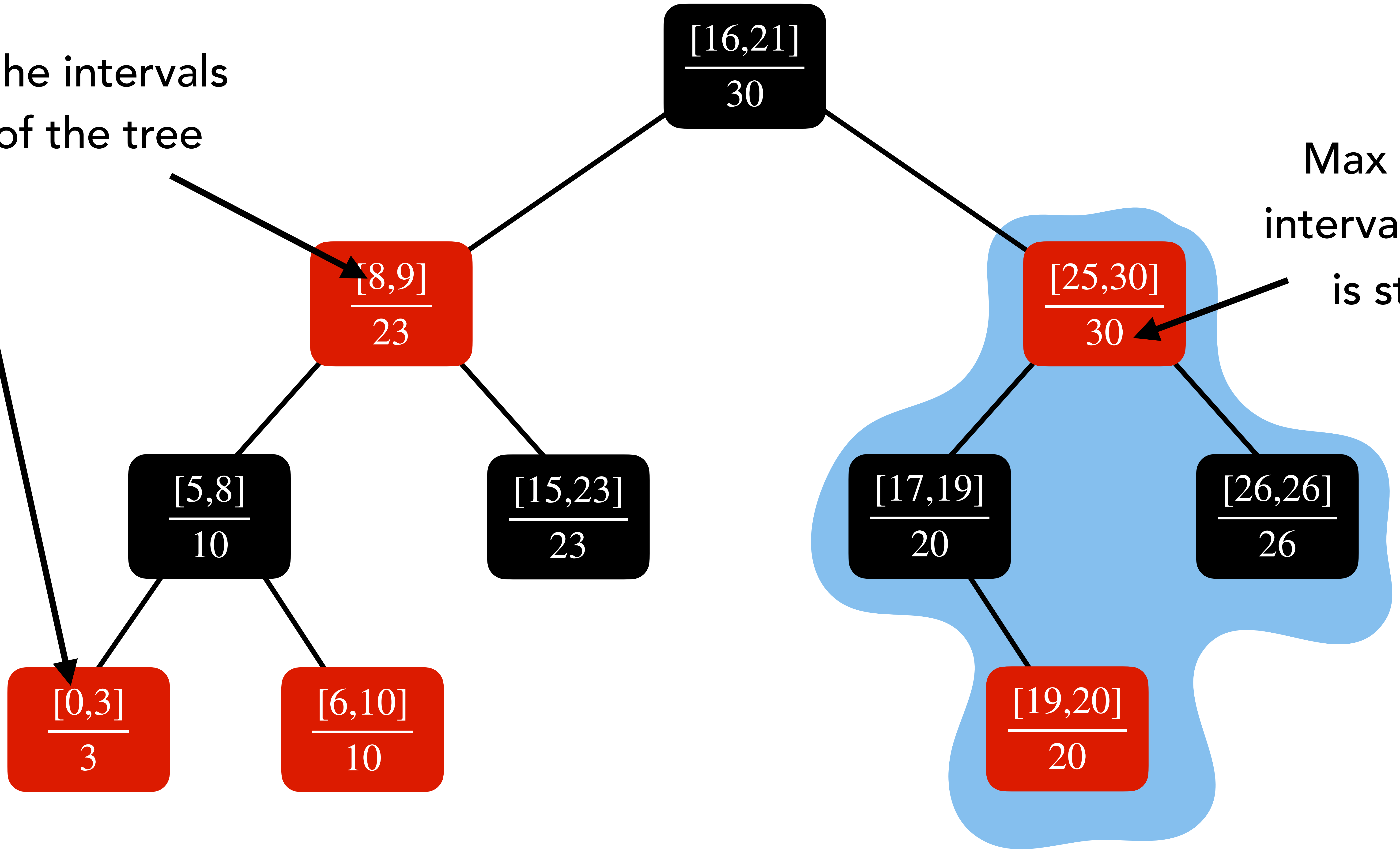

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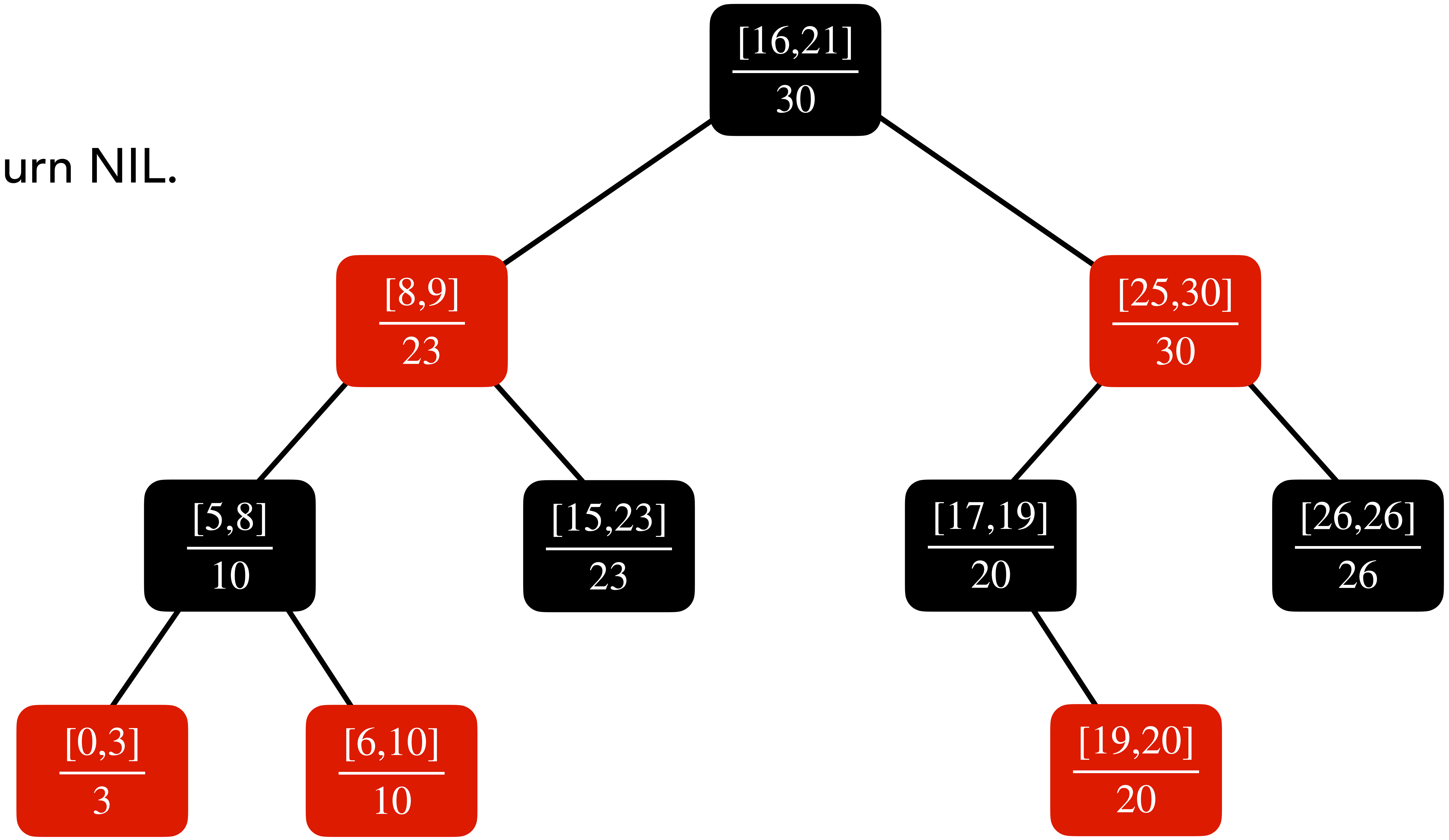
- The key of x is $x.int.low$.
- Maximum of all the high values of the intervals in the **subtree**(x) is stored as $x.max$.

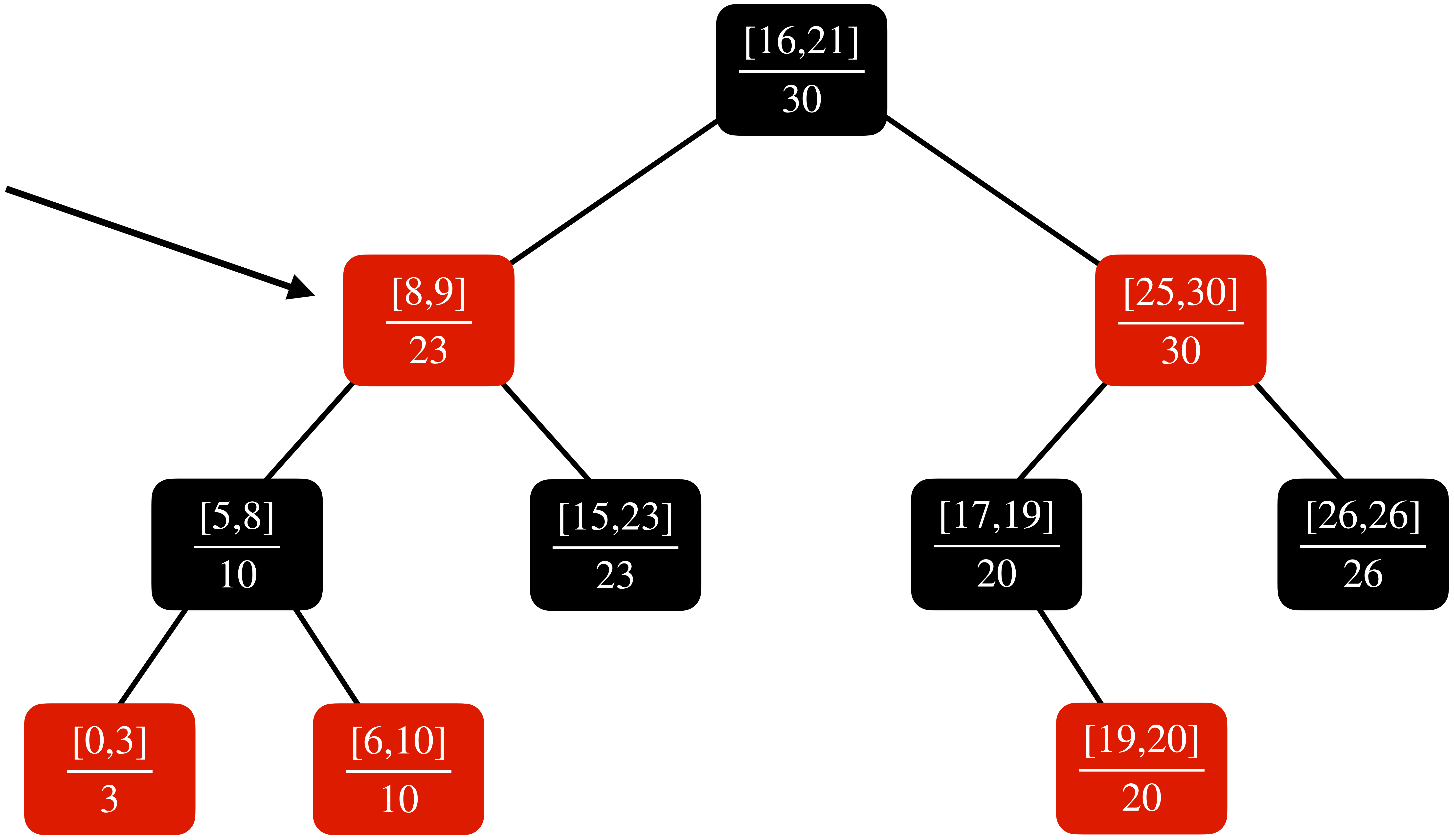
Idea Behind Interval-Search

Interval-Search$(T, [11{,}14])$ should return NIL.



Idea Behind Interval-Search

Interval-Search$(T, [6,10])$ can return



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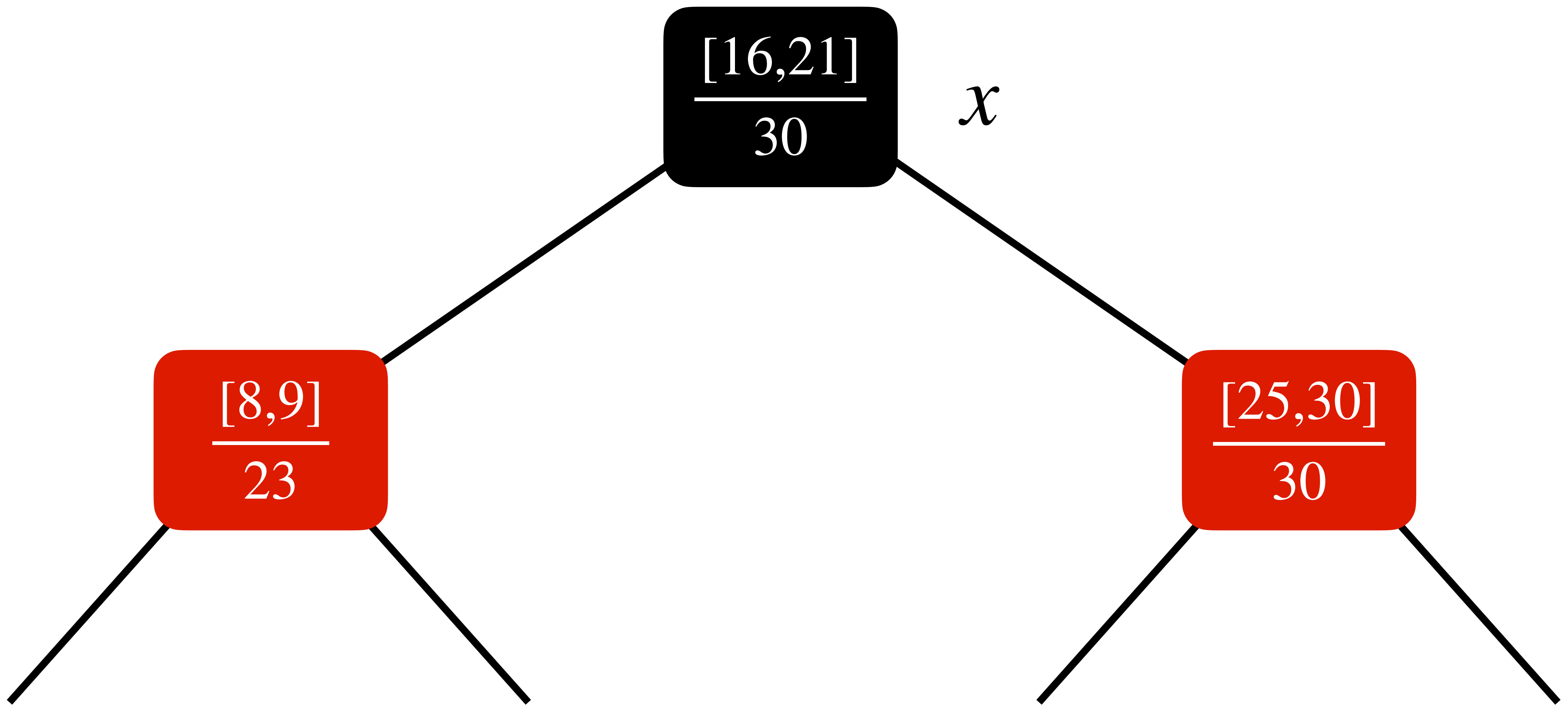

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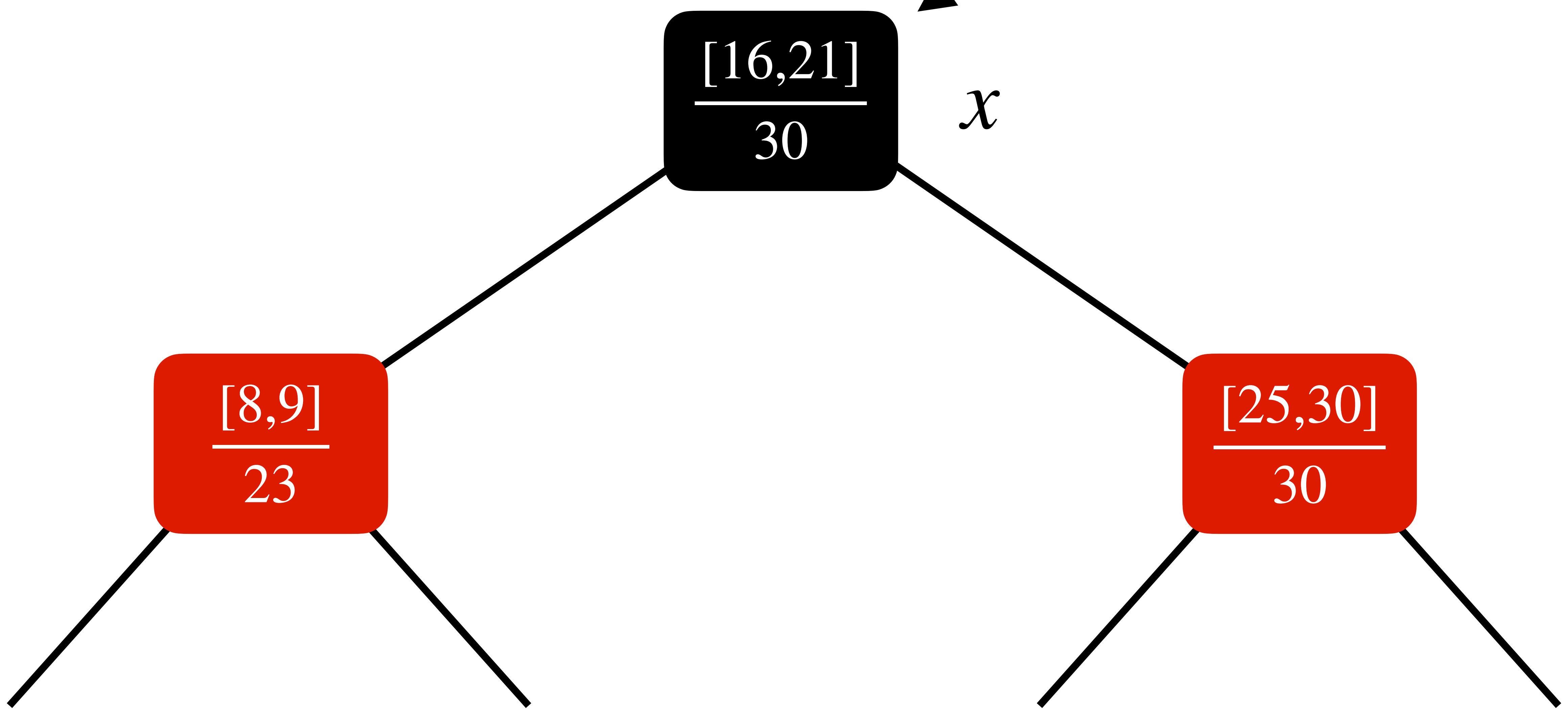
Find the node with interval overlapping with $i = [25,40]$.

[16,21] clearly doesn't overlap with [25,40].



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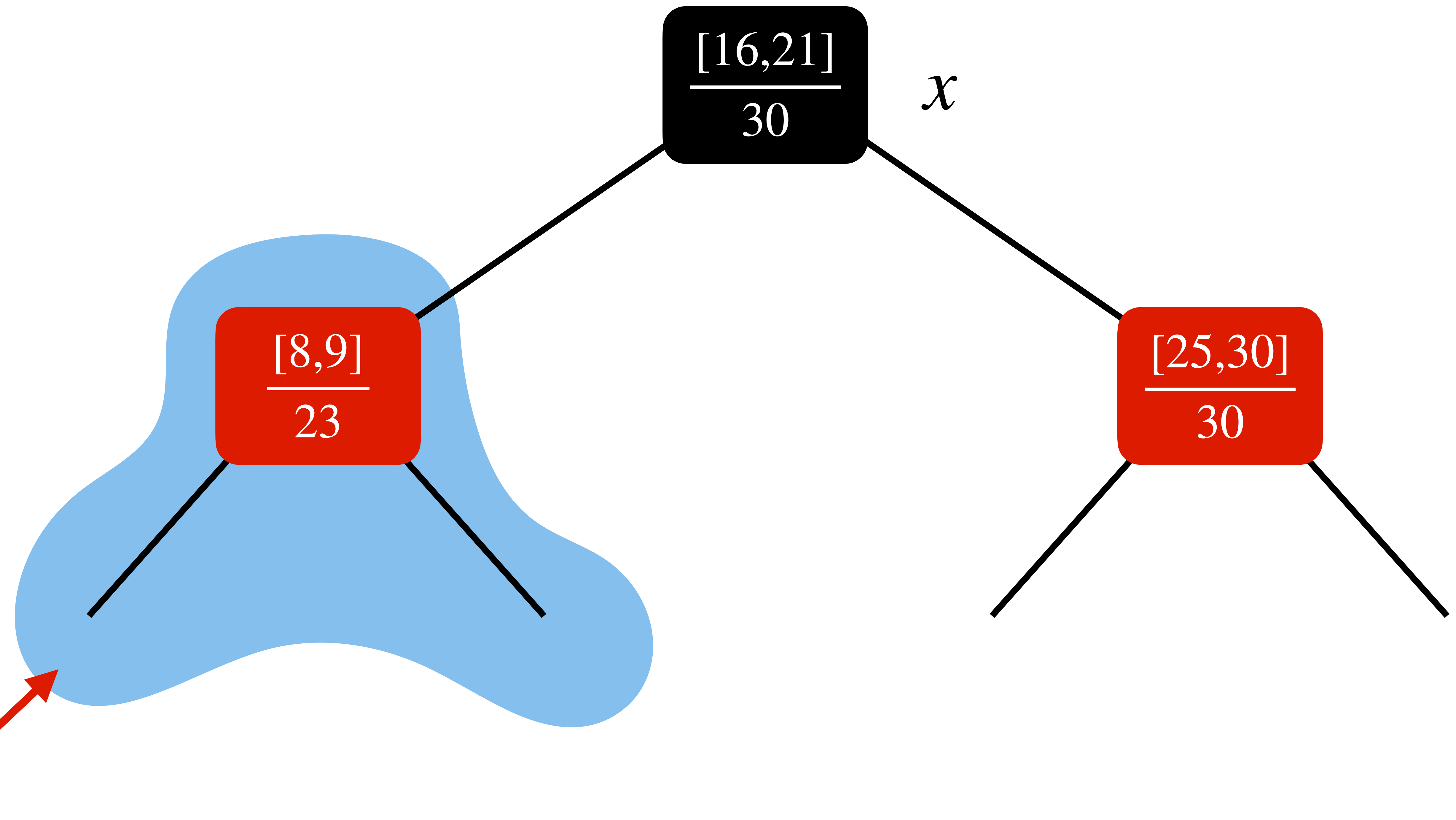

Can [25,40] overlap with an interval in the left subtree?

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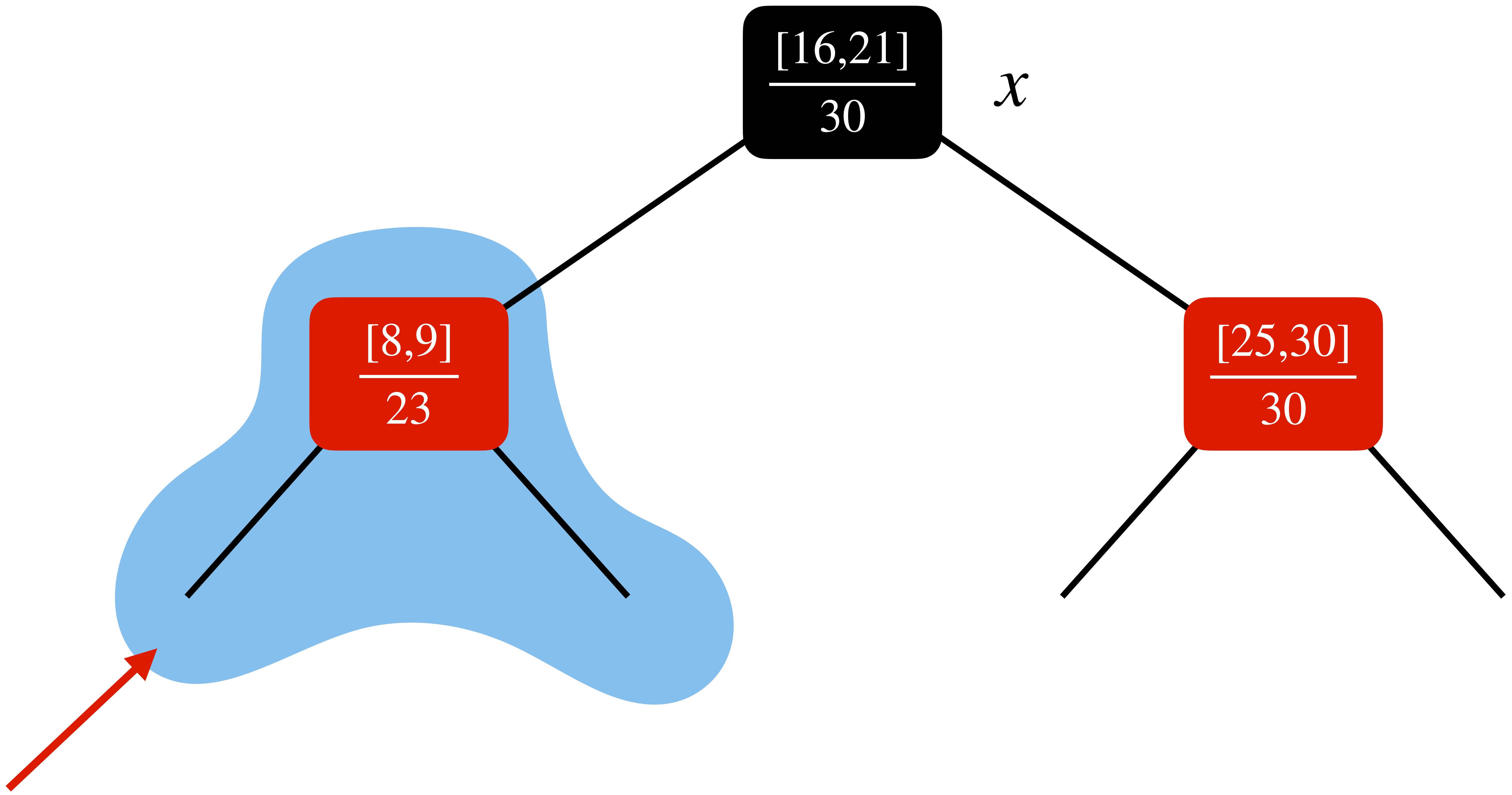

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No, because maximum high is 23.

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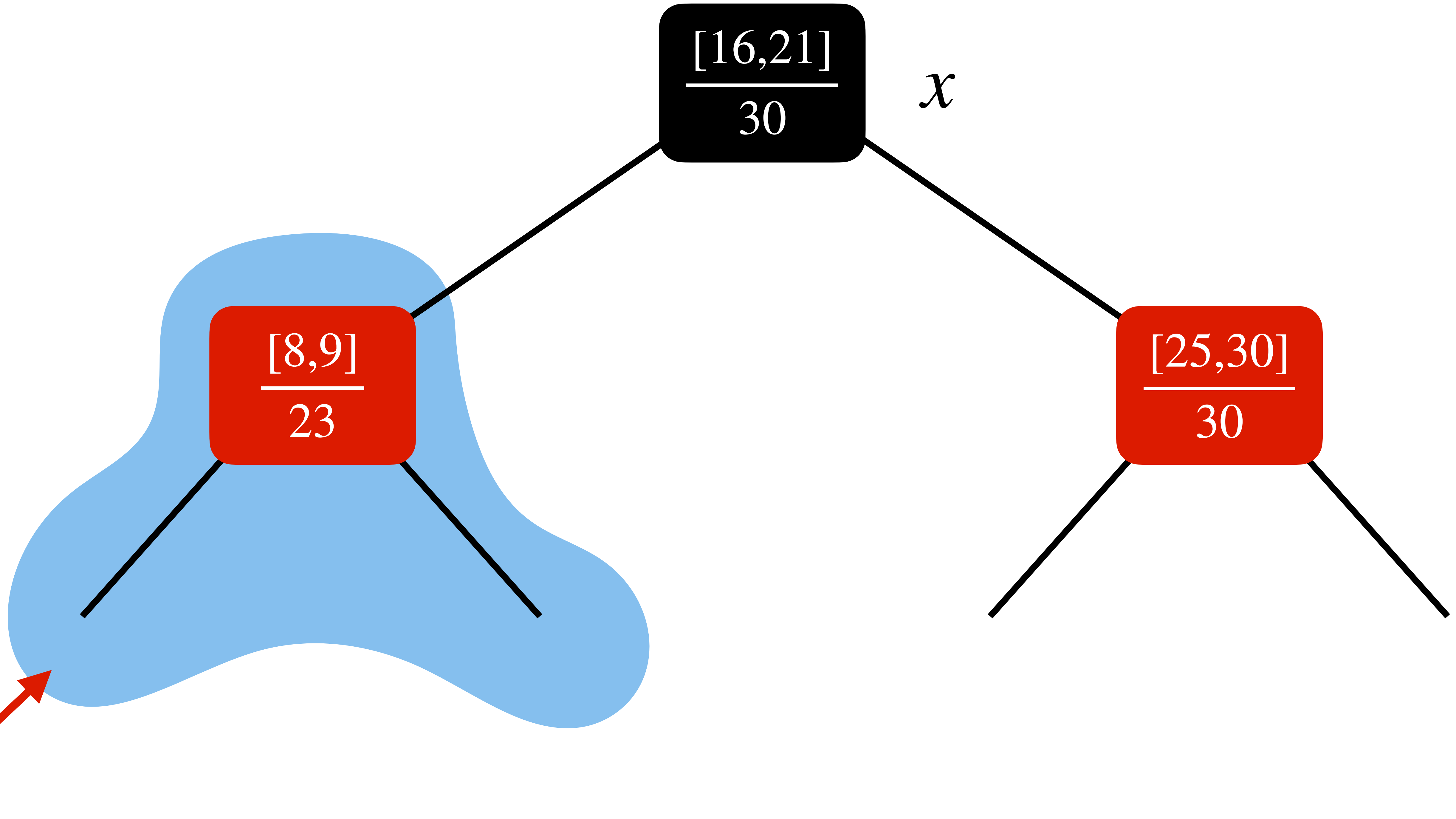

Can $[25,40]$ overlap with an interval in the left subtree?

No, because maximum high is 23. Hence, every interval in this subtree will be to the left of $[25,40]$.

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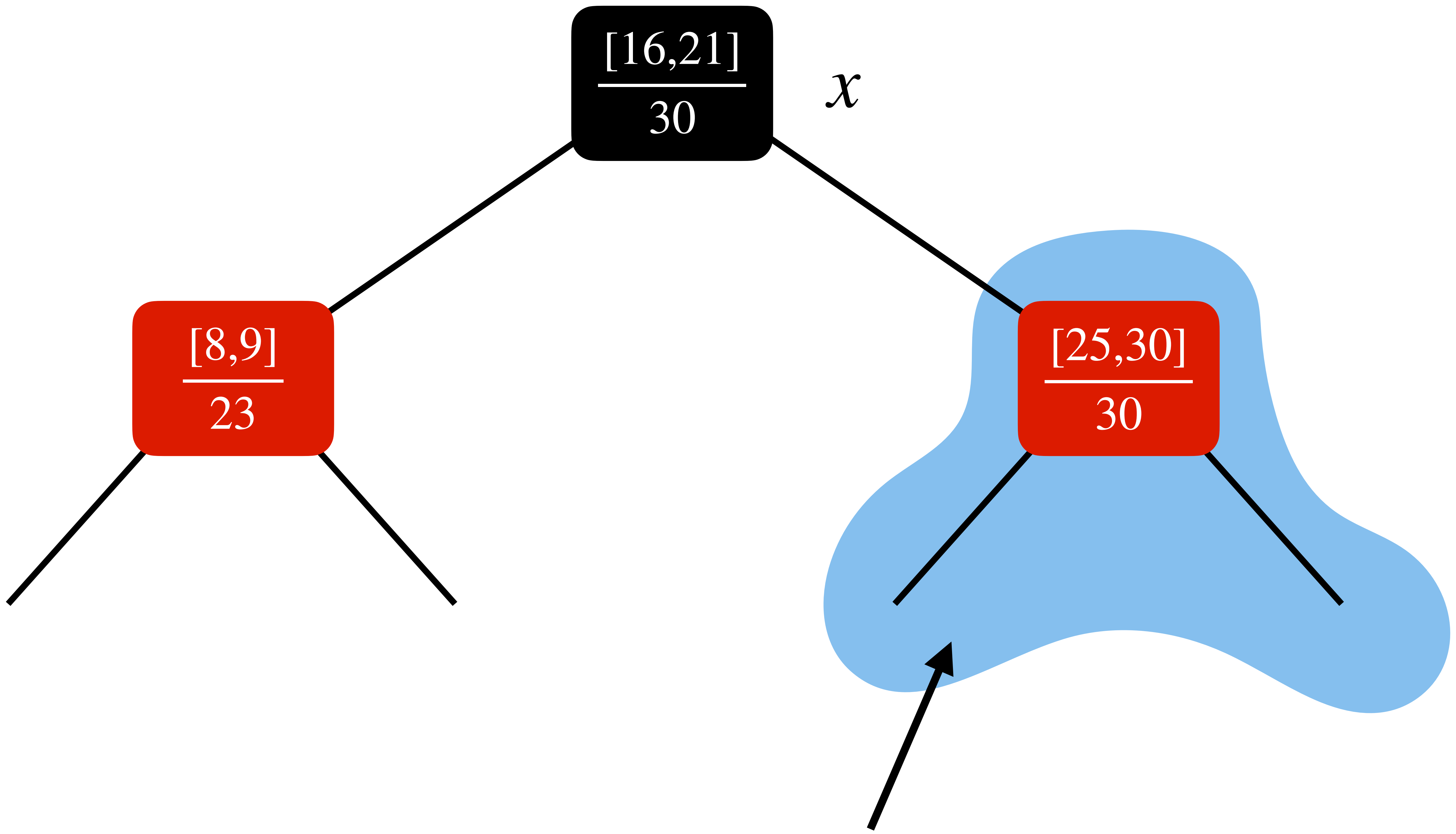

Any interval overlapping with $[25,40]$, if present must be in the right subtree.

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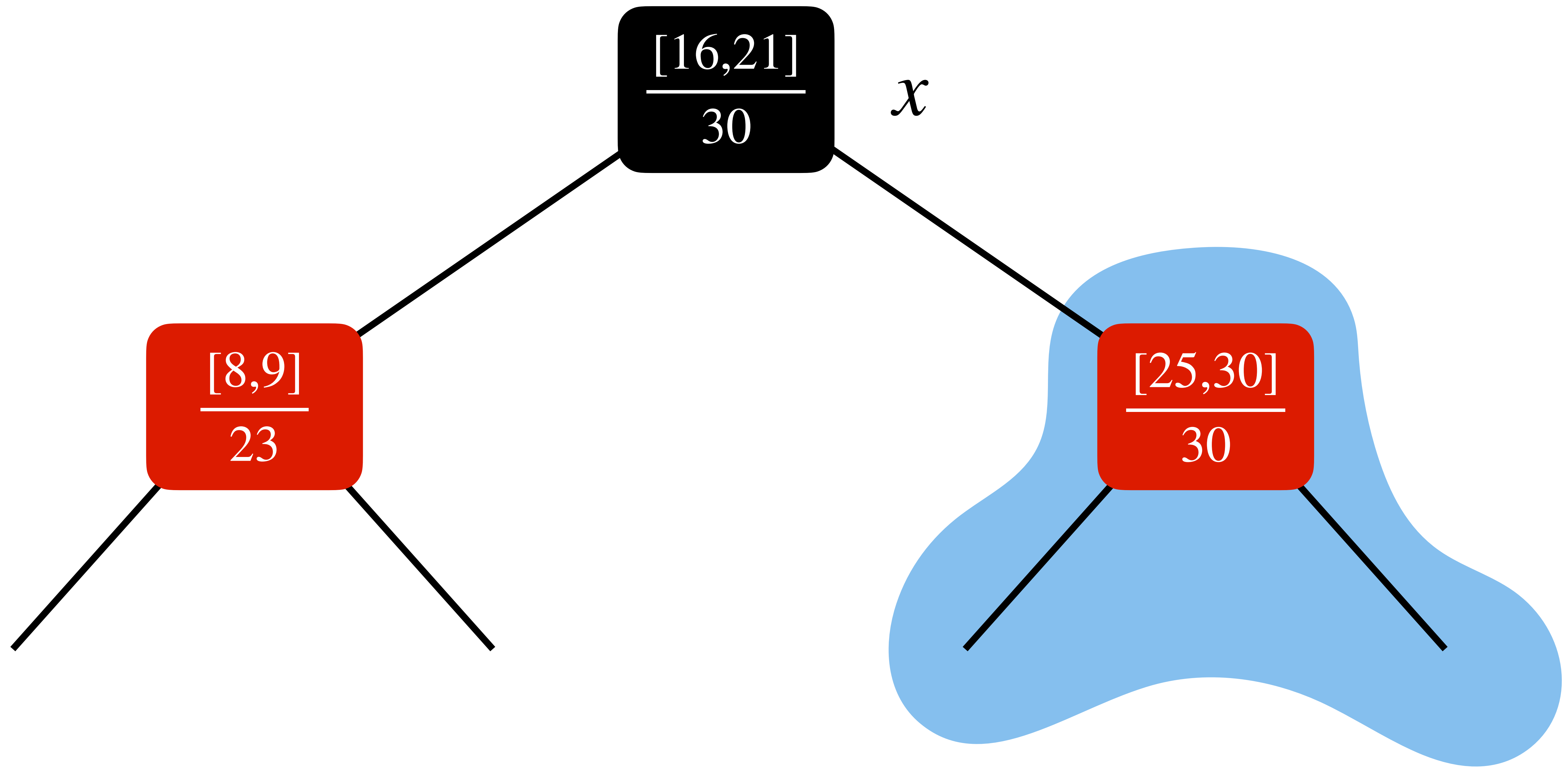
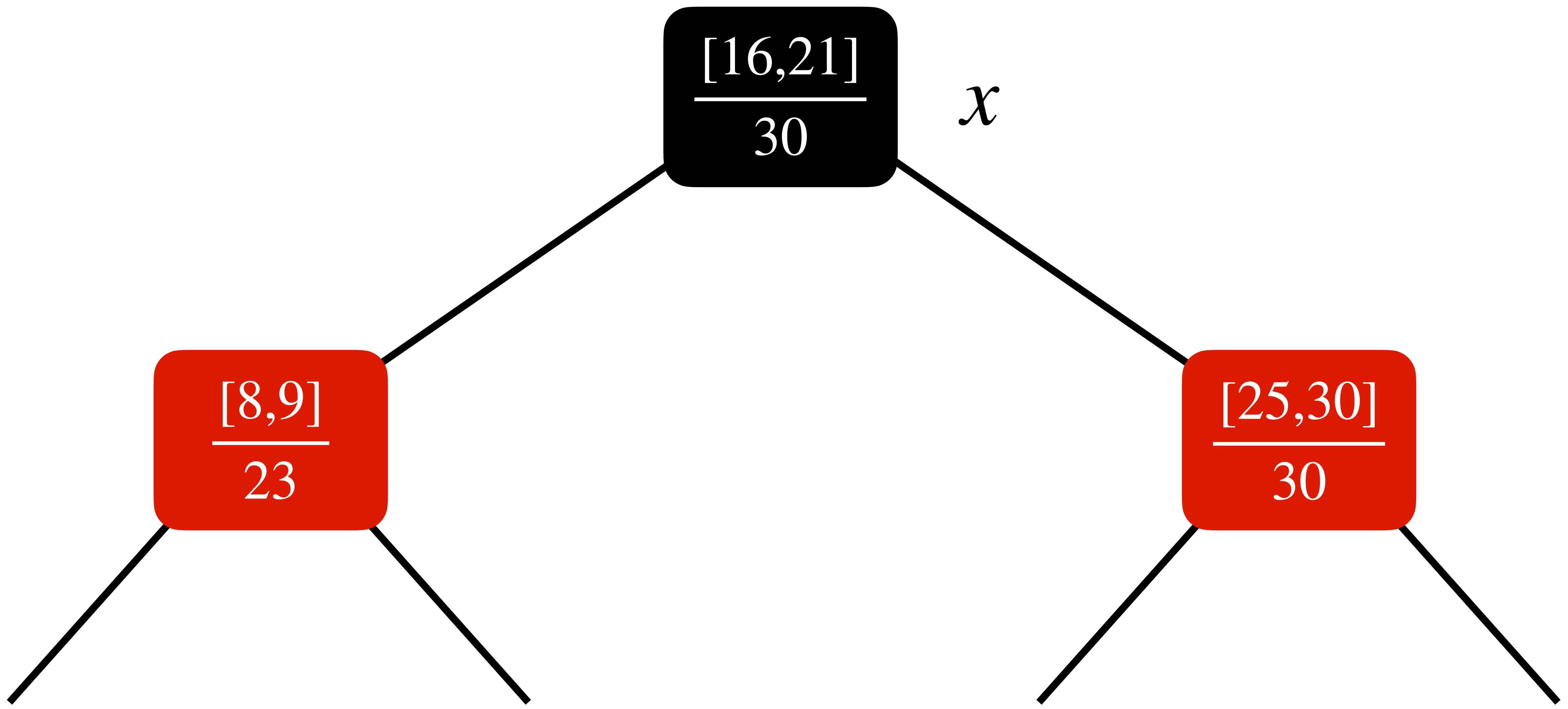


When $i.low > x.left.max$, go right

Idea Behind Interval-Search

Find the node with interval overlapping with $i = [10,14]$.



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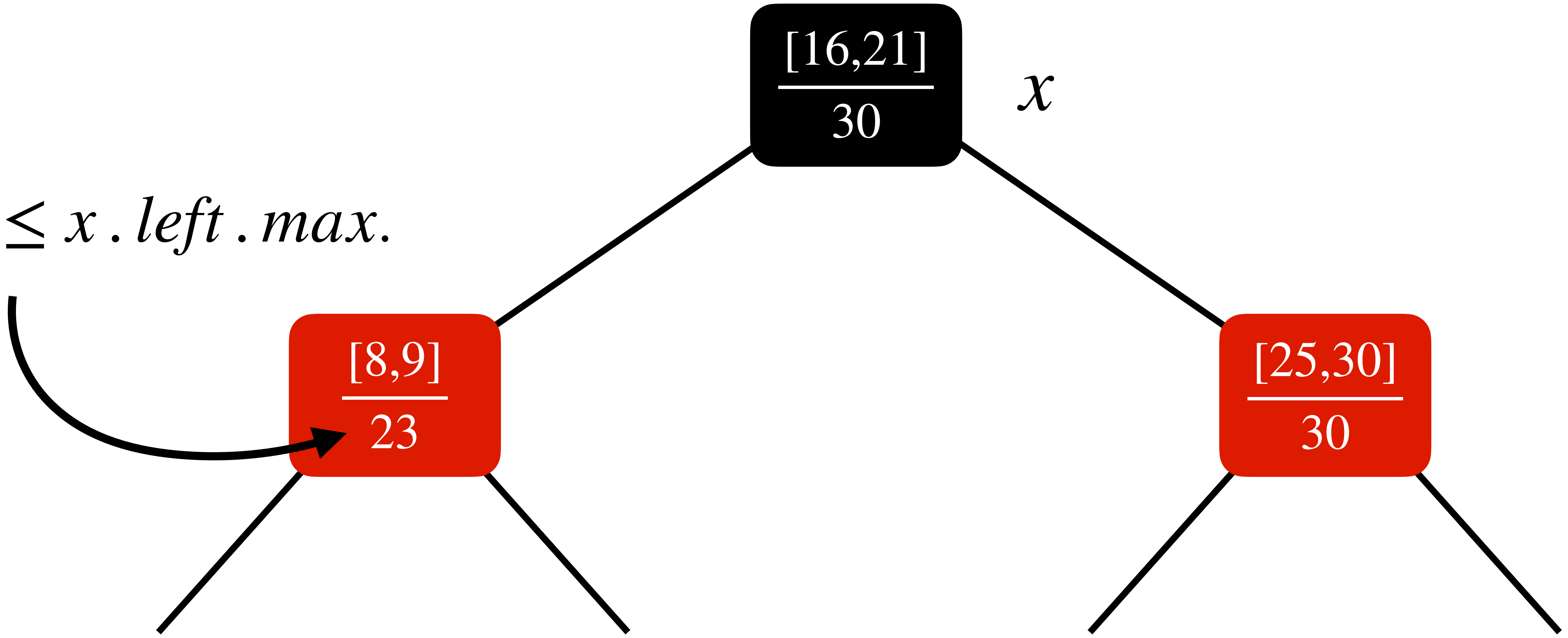
Find the node with interval overlapping with $i = [10,14]$.

Notice that $i.low \le x.left.max$.



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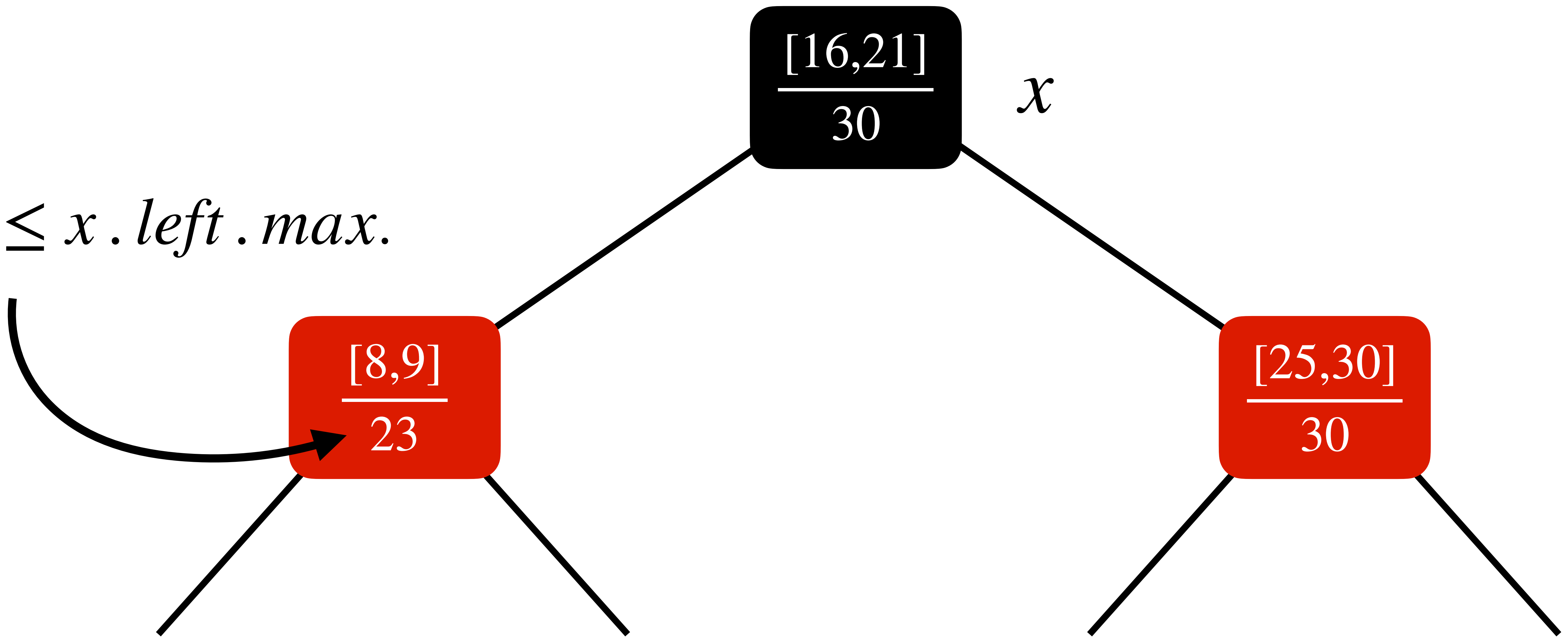

We will prove now that when $i.low \leq x.left.max$, it is safe to go left.

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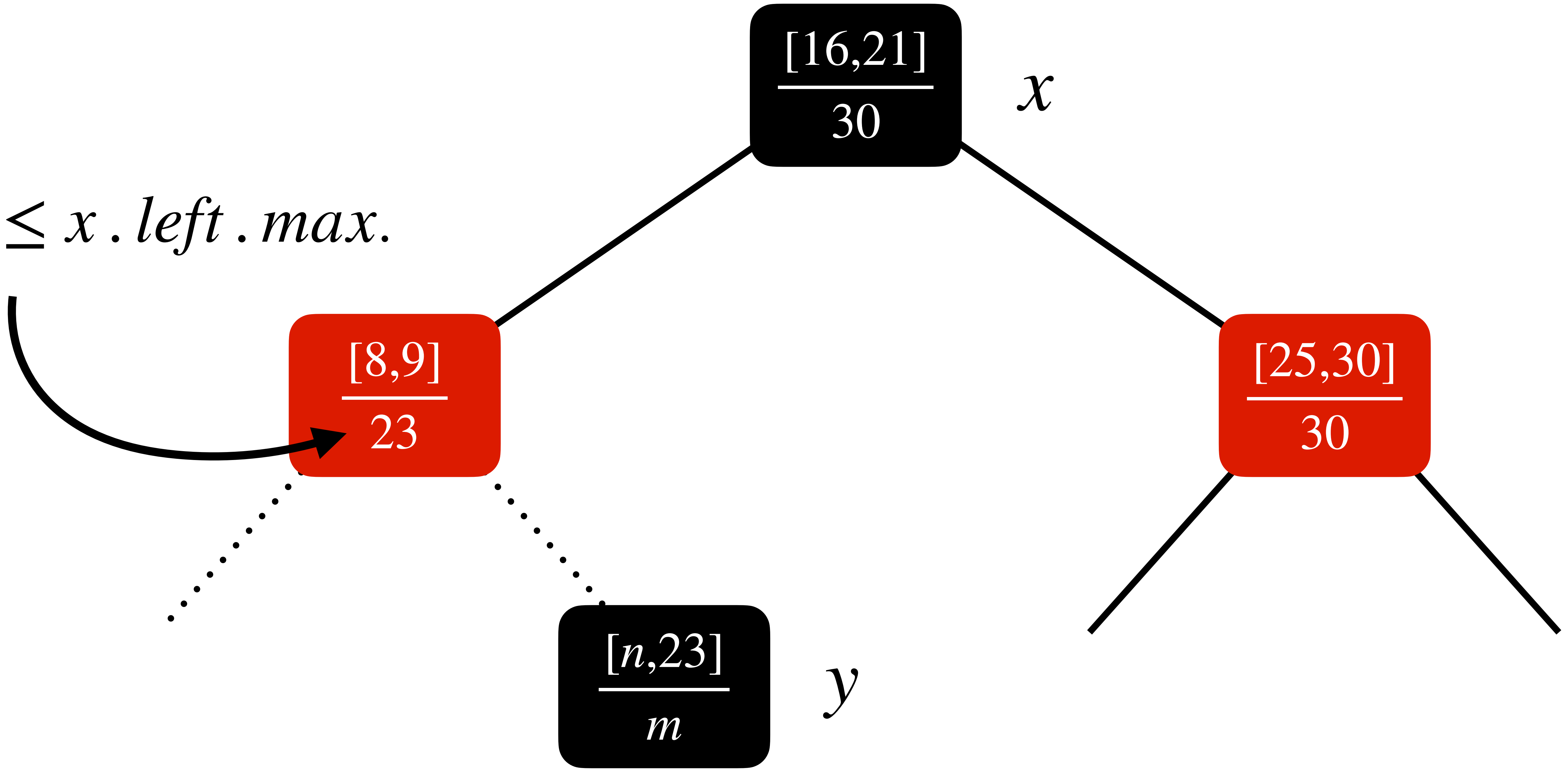

There must exist a node like this in the left subtree

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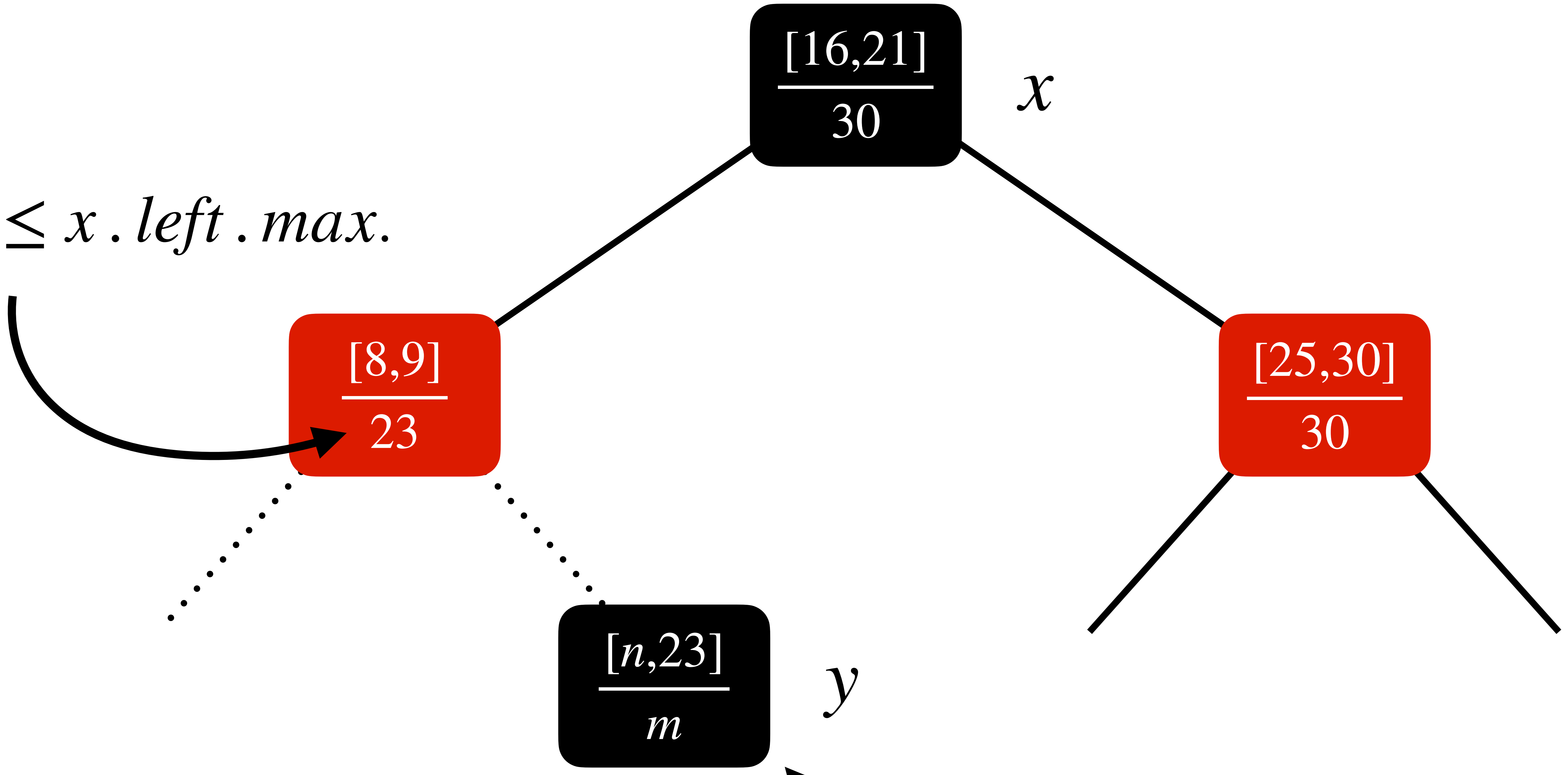

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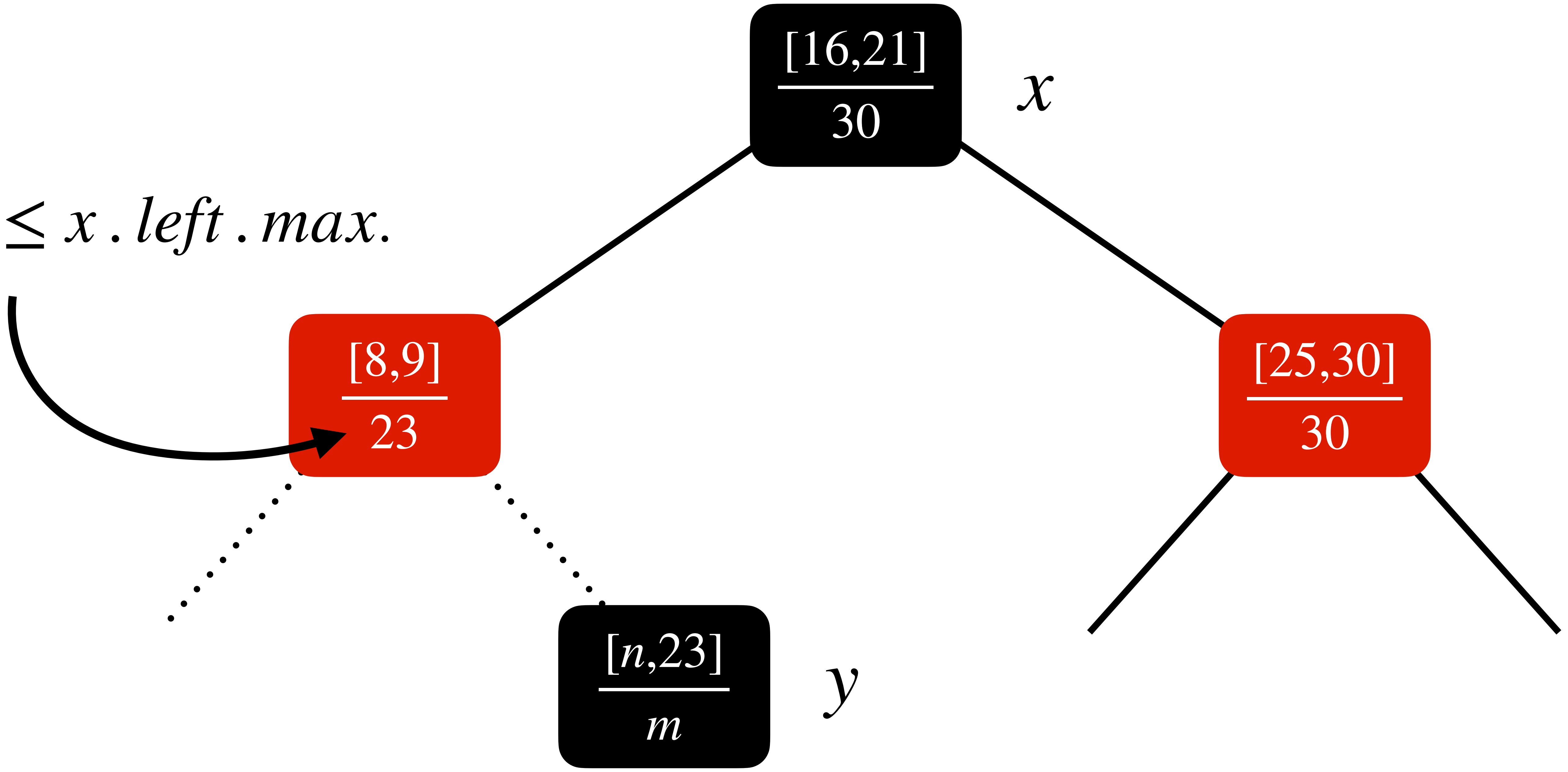

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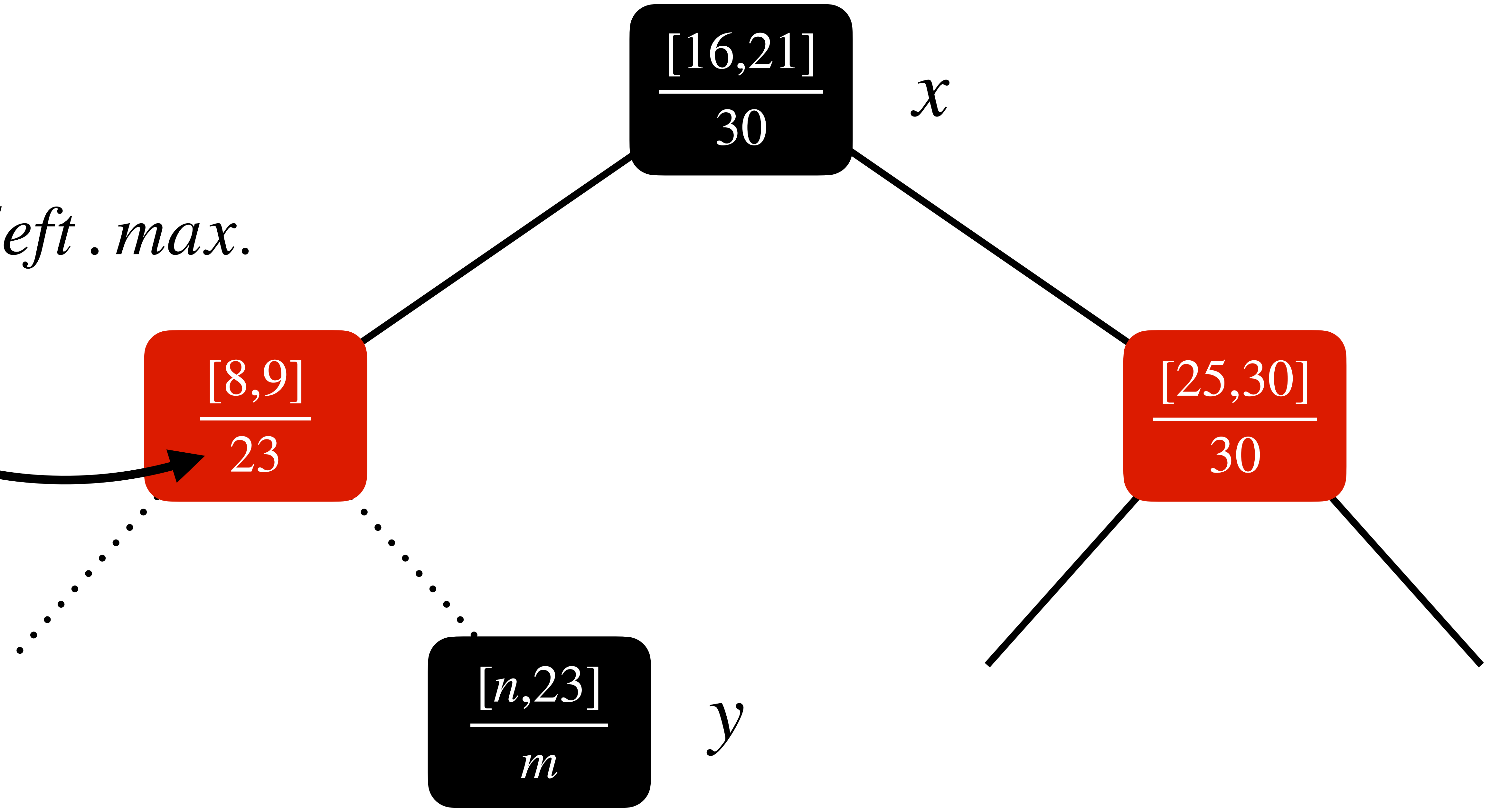

If $n \leq 14$, y in the left subtree will overlap with $[10,14]$.

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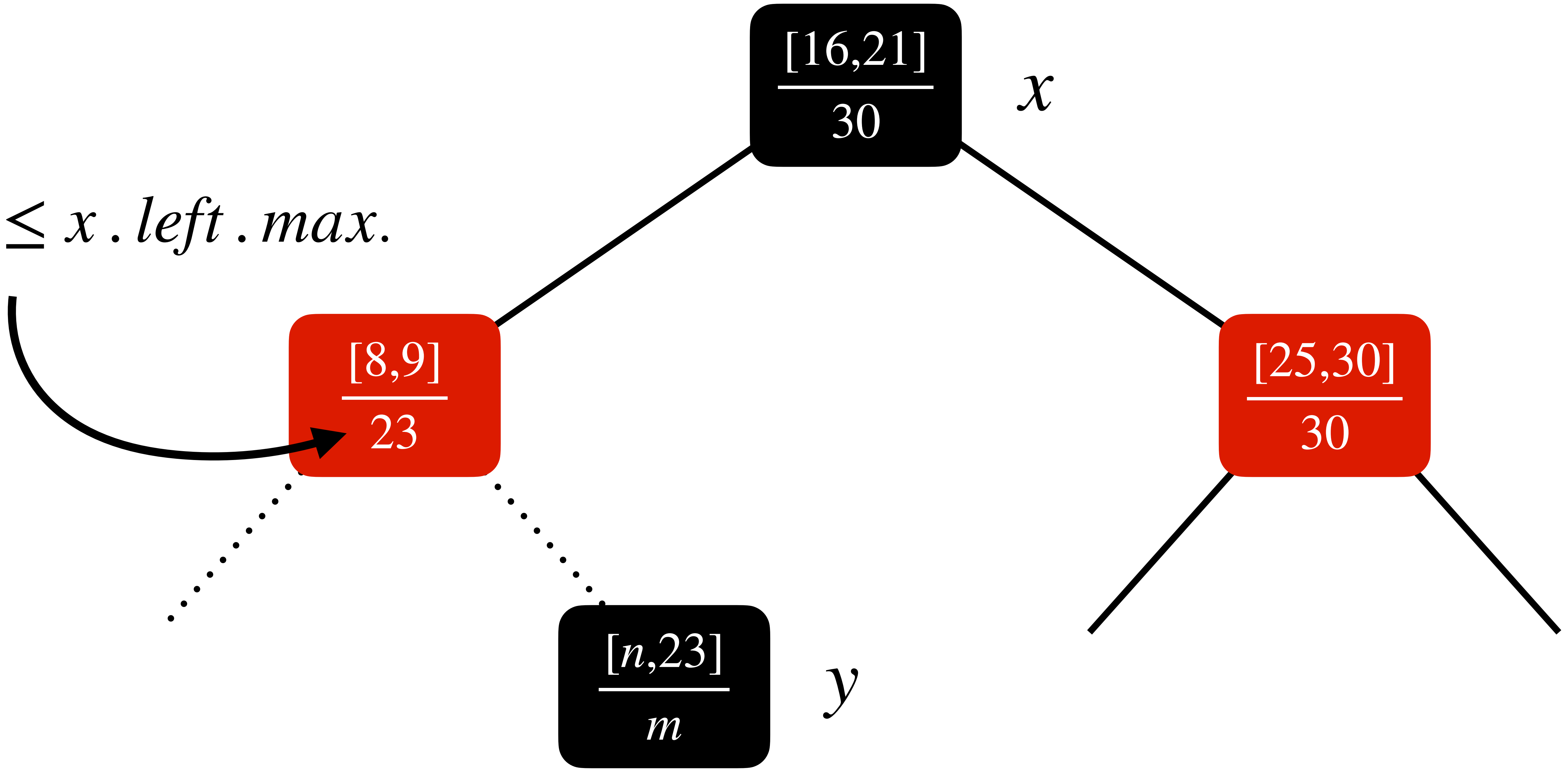

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If $n \leq 14$, y in the left subtree will overlap with $[10,14]$.

If $n > 14$, y will not overlap with $[10,14]$.

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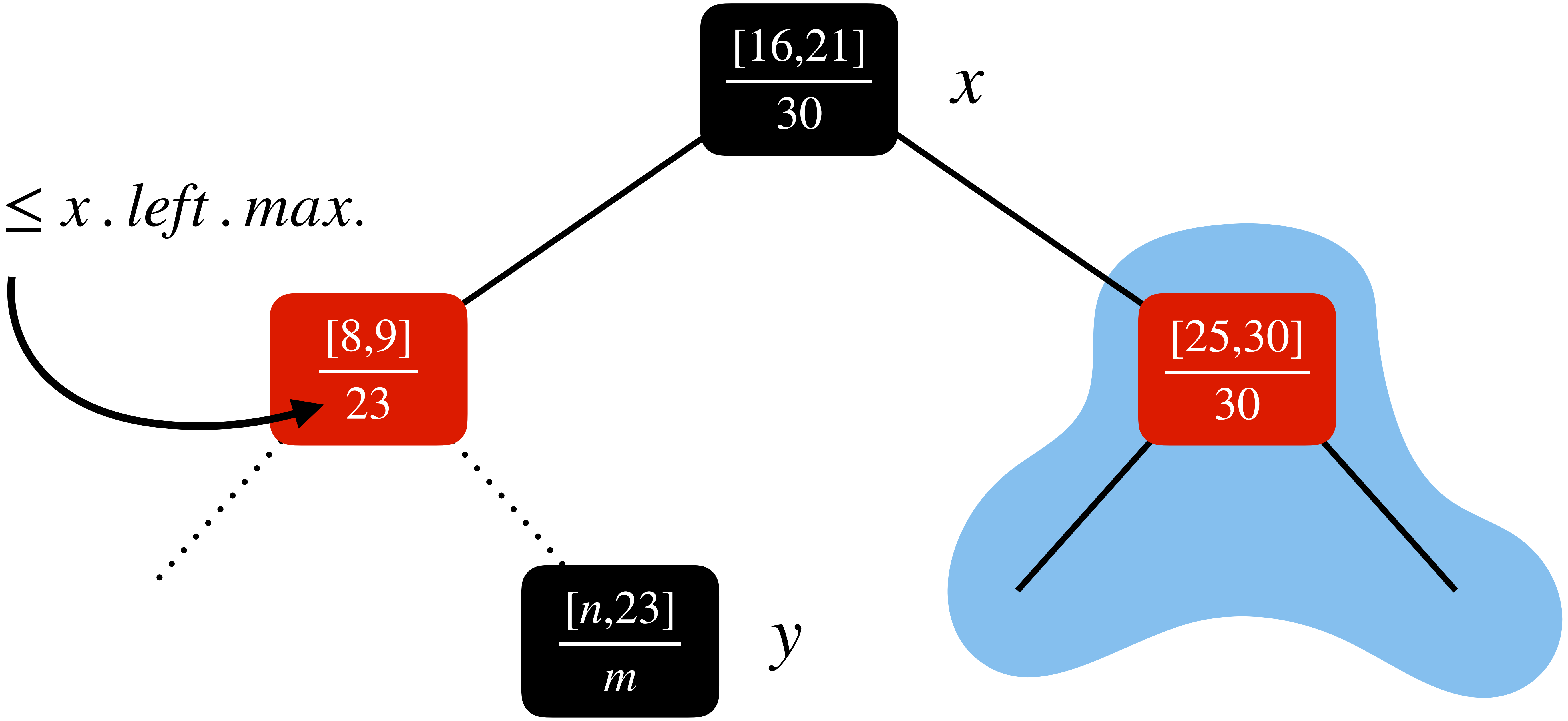

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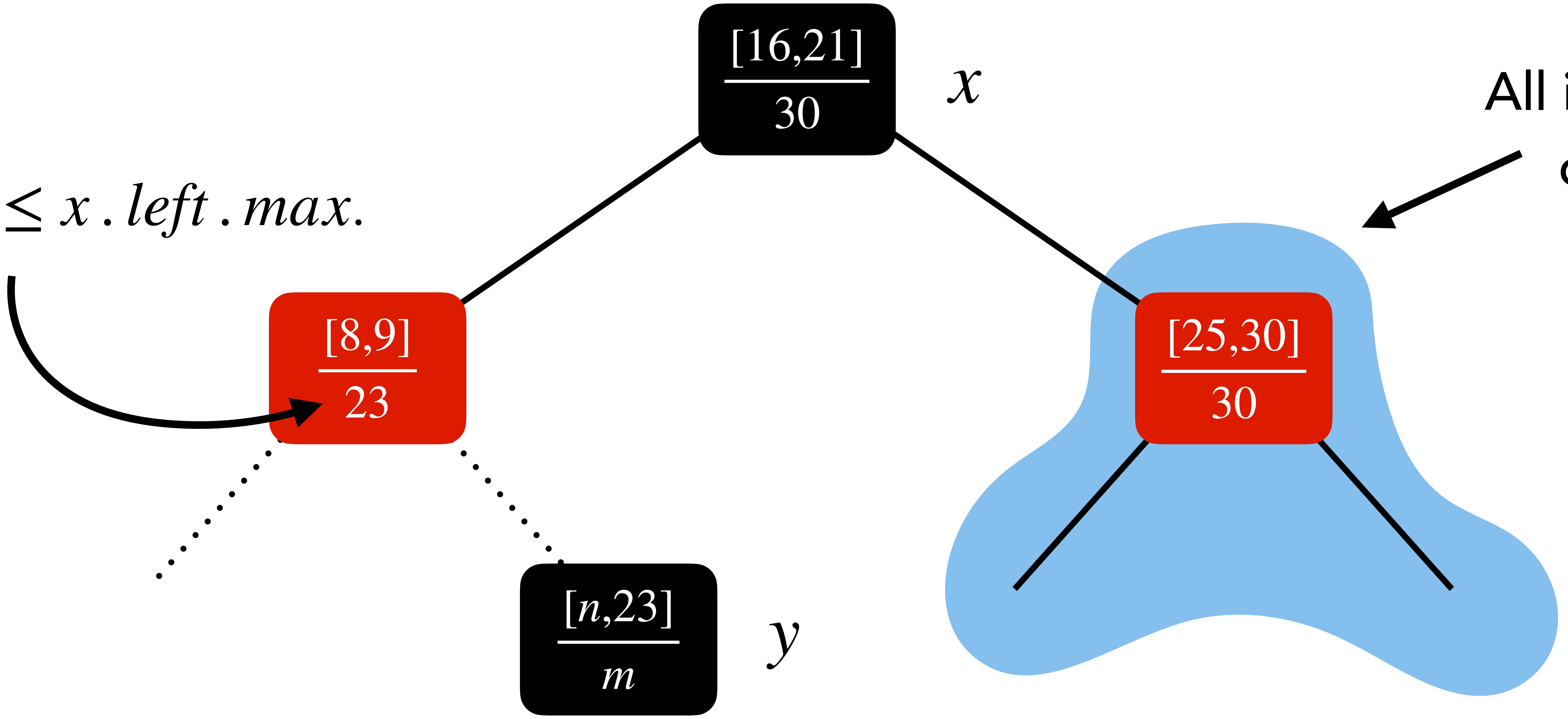

Notice that $i.low \leq x.left.max$.

All intervals' low will be > 14 due to BST properties.

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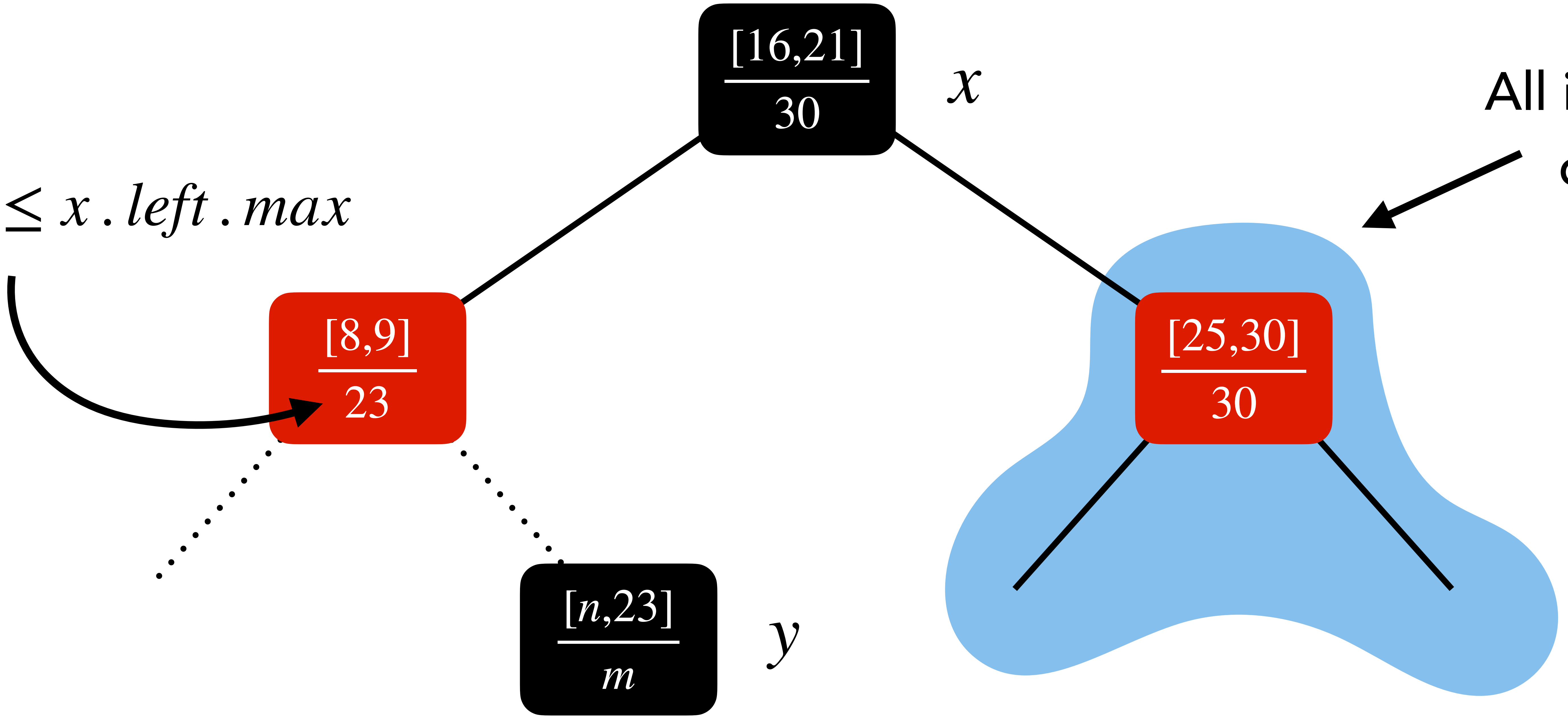

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All intervals' low will be > 14 due to BST properties.

If $n \leq 14$, y in the left subtree will overlap with $[10,14]$.

If $n > 14$, no node in the right subtree will overlap with $[10,14]$.

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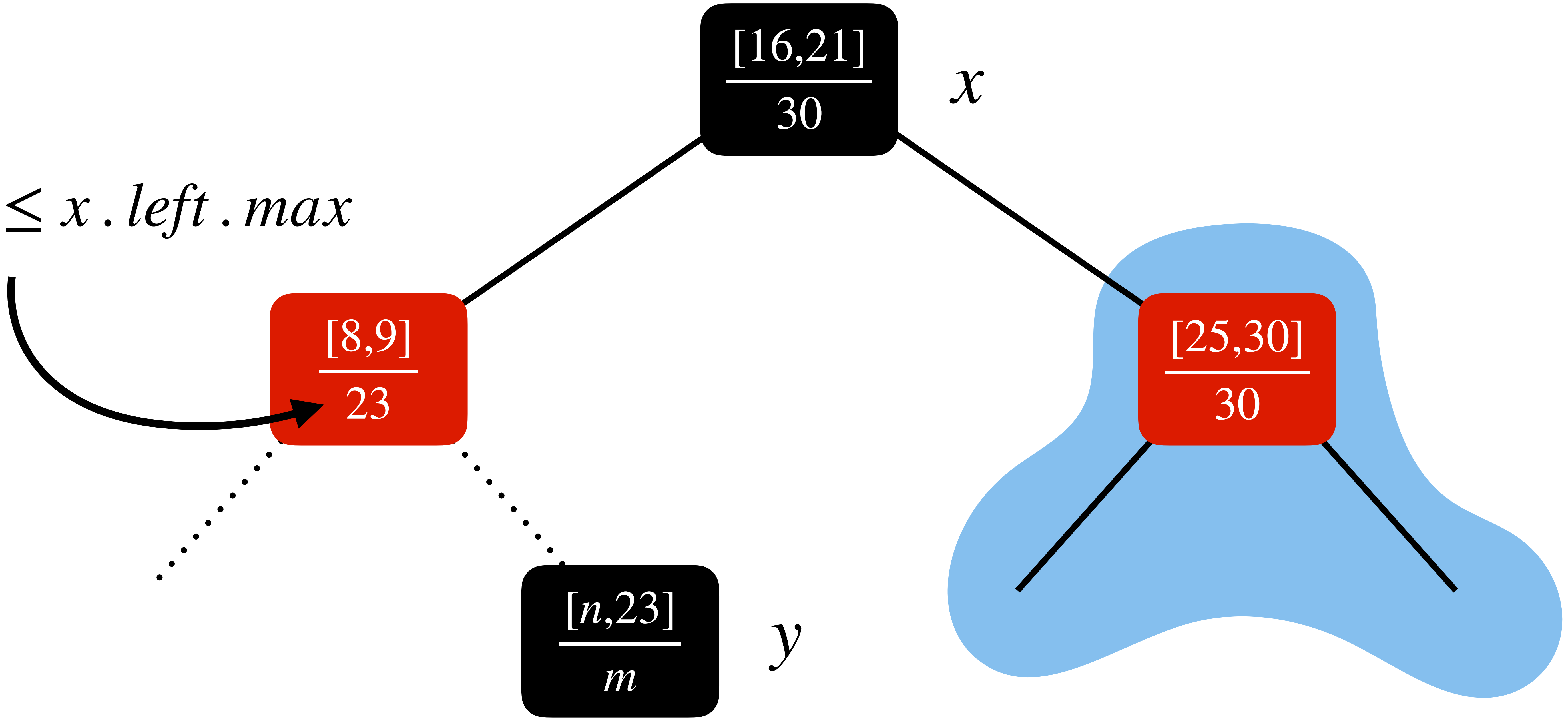

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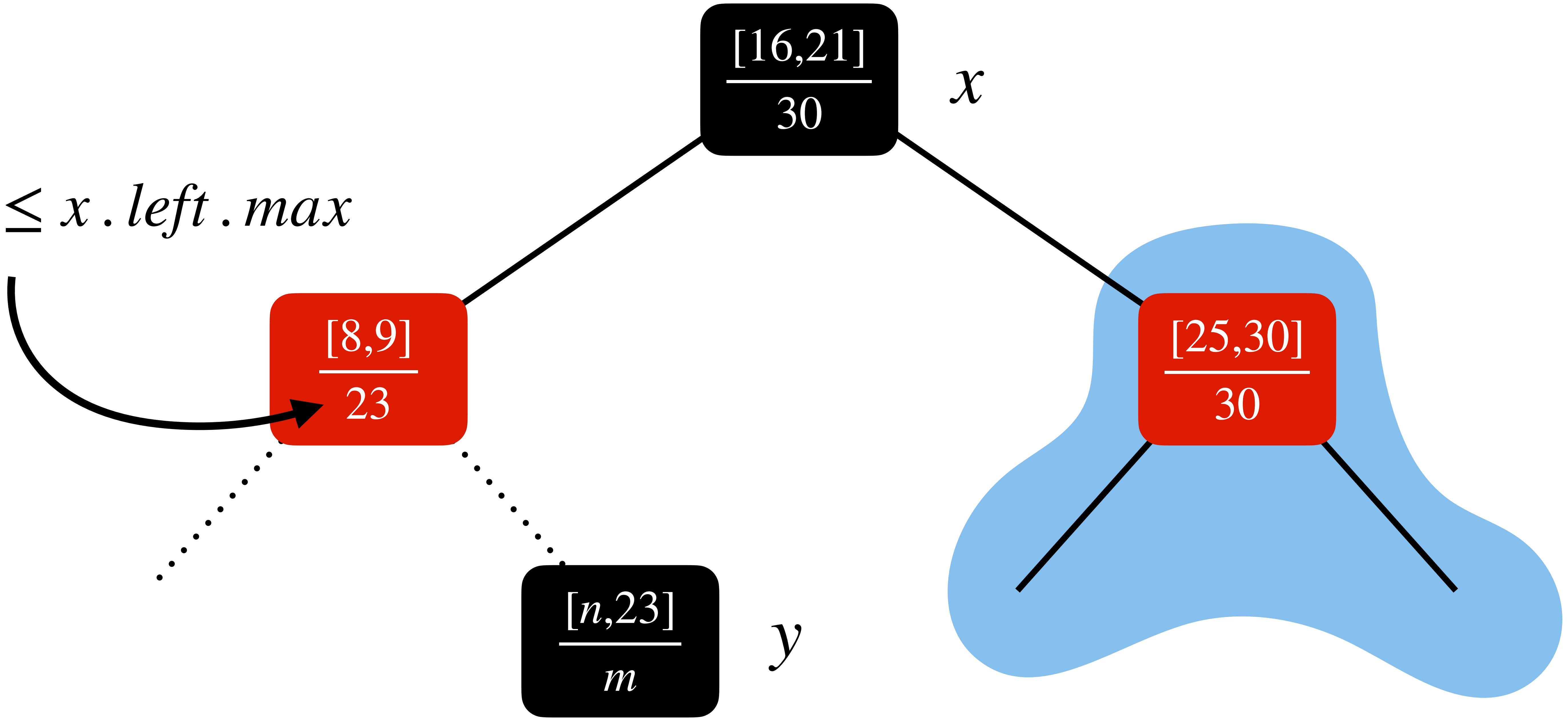

When $i.low \le x.left.max$, go left

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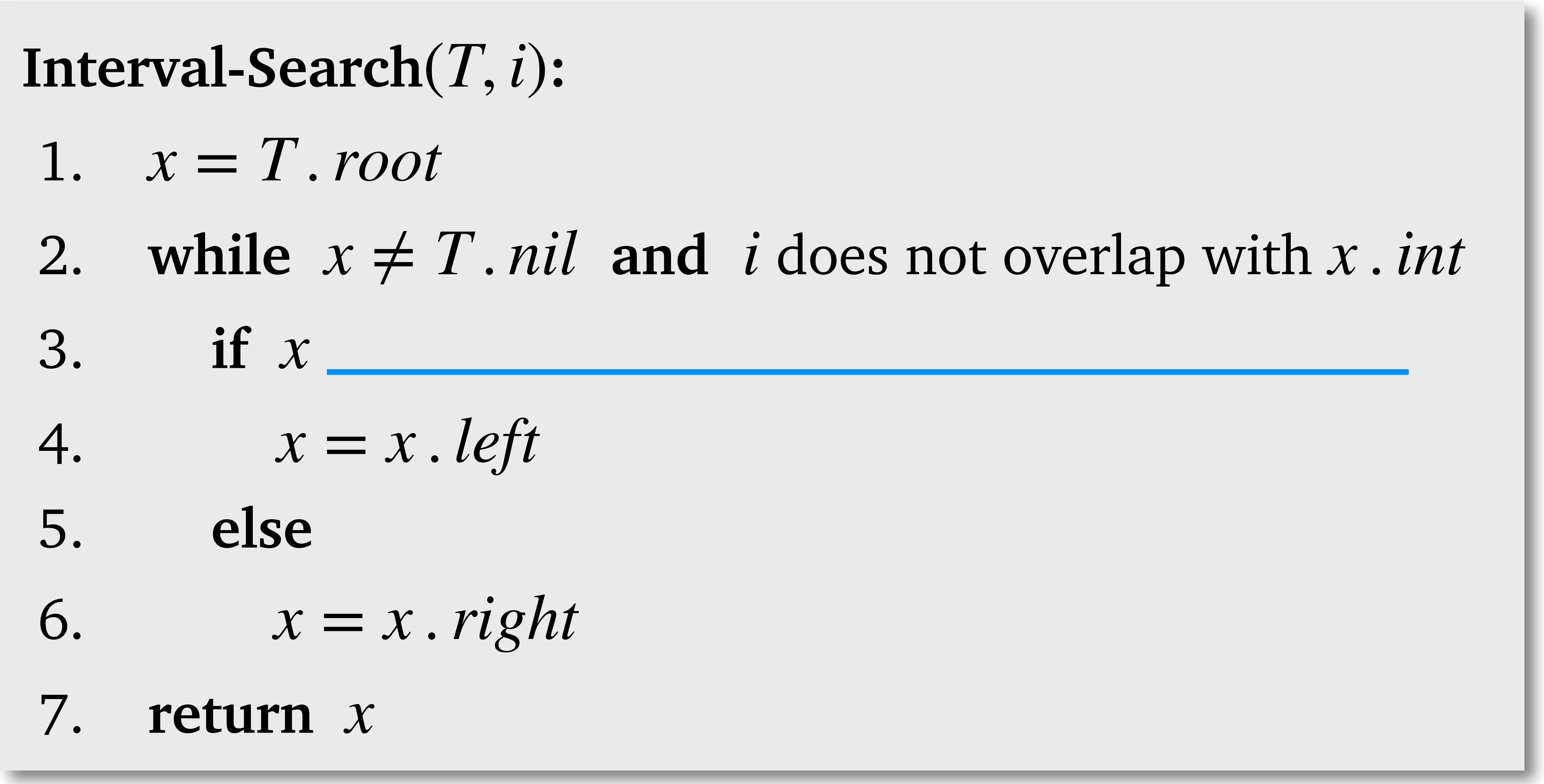

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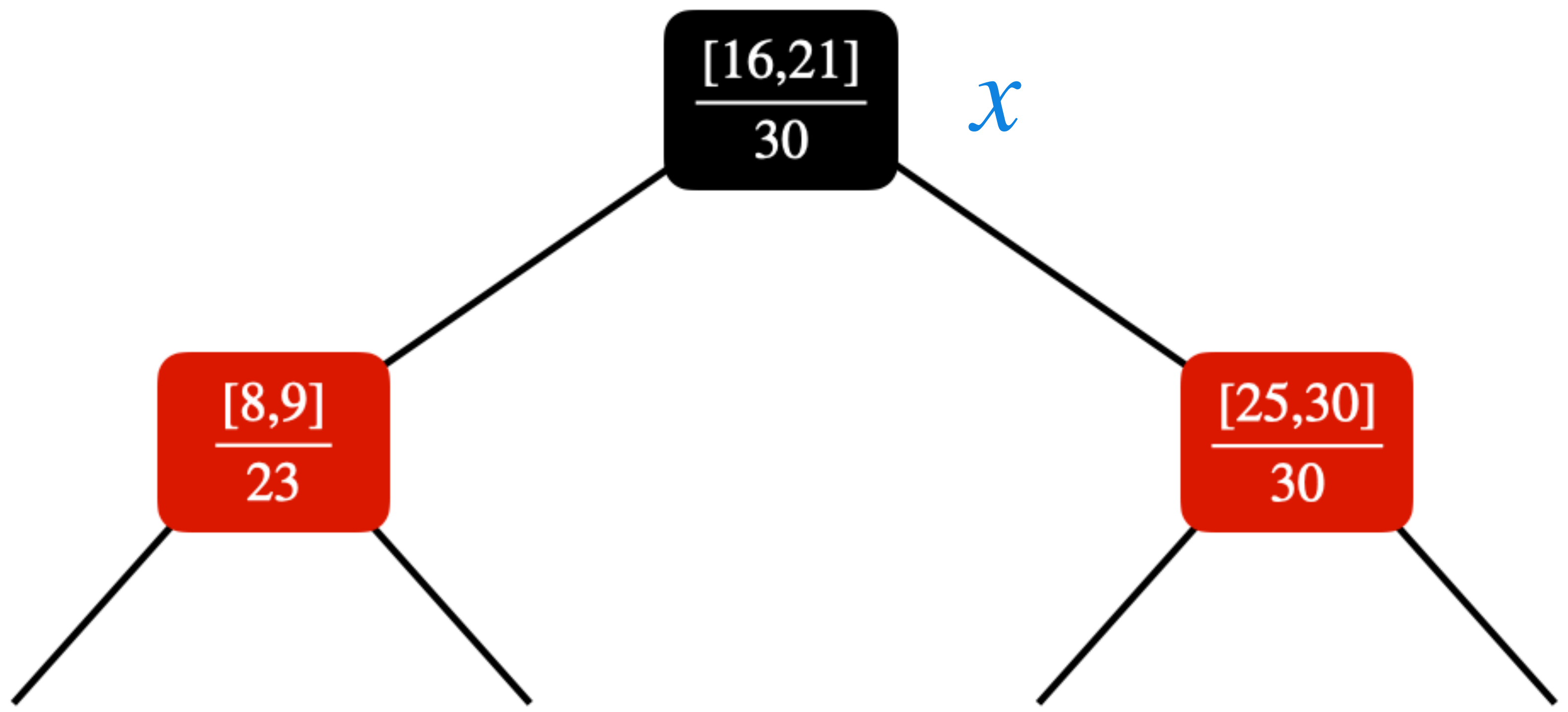
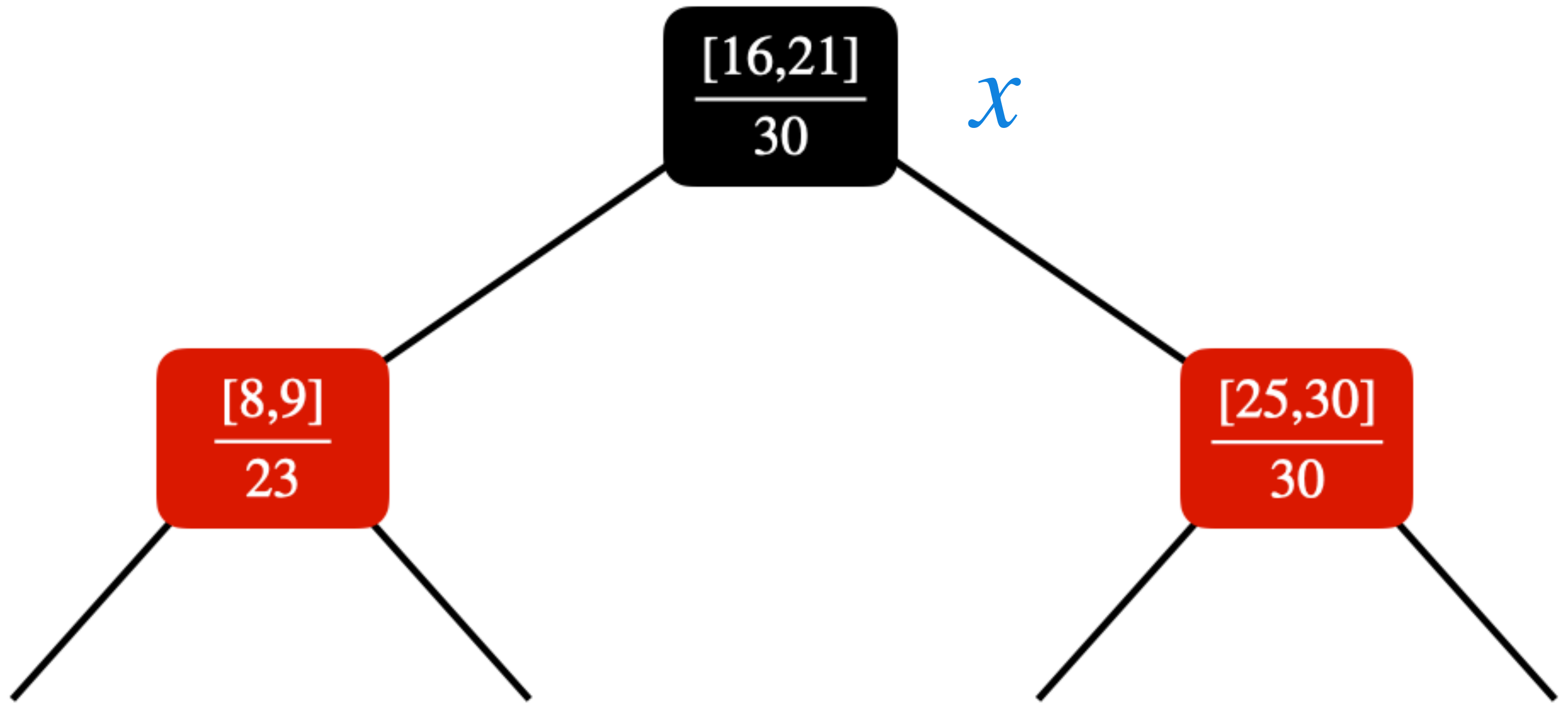


Interval-Search Pseudocode

```
Interval-Search($T, i$):
    $x = T.root$
    while $x \neq T.nil$ and $i$ does not overlap with $x.int$
        if $x.left \neq T.nil$ and $x.left.max \geq i.low$
            $x = x.left$
        else
            $x = x.right$
    return $x$
```

$i = [10,14]$



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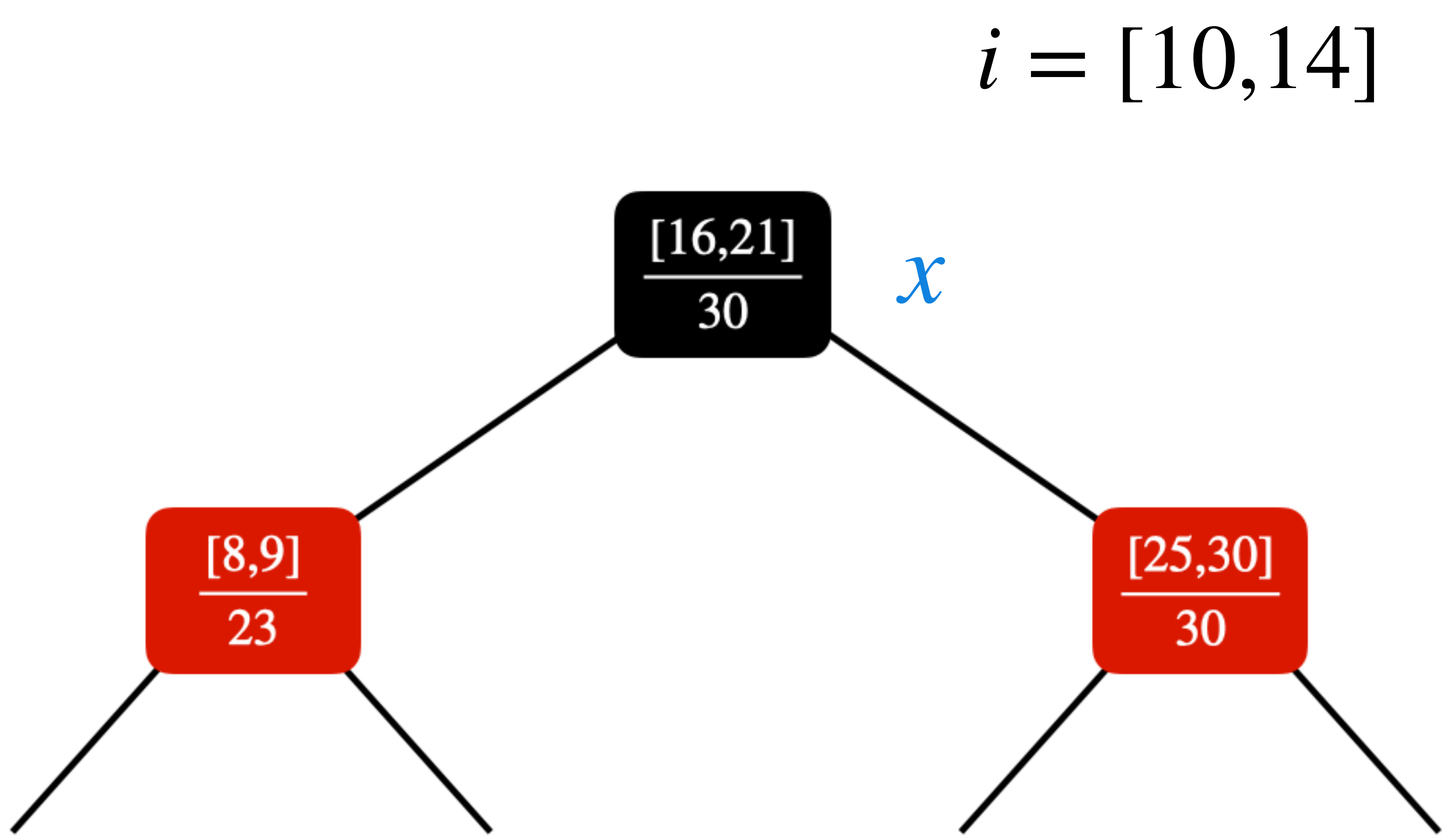
Interval-Search($T, i$):
    $x = T.root$
    while $x \neq T.nil$ and $i$ does not overlap with $x.int$
        if $x.left \neq T.nil$ and $x.left.max \geq i.low$
            $x = x.left$
        else
            $x = x.right$
    return $x$
```



Time Complexity: $\Theta(h) = \Theta(\log n)$ as with every iteration algorithm goes one level down.

Maintaining Subtree Max Highs

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Idea: Inserting or deleting an element will only affect the maximum values of its ancestor.

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Similar to how we maintained sizes of the subtrees in the previous data structure.